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UPPER QUASI CONTINUOUS MAPS AND QUASI CONTINUOUS SELECTIONS

Milan Matejdes, Zlín

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Abstract. The paper deals with the existence of a quasi continuous selection of a multifunction for which upper inverse image of any open set with compact complement contains a set of the form $(G \setminus I) \cup J$, where G is open and I, J are from a given ideal. The methods are based on the properties of a minimal multifunction which is generated by a cluster process with respect to a system of subsets of the form $(G \setminus I) \cup J$.

Keywords: selection, quasi continuity, minimal usco multifunction, cluster point, generalized quasi continuity

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1. Introduction

The upper and lower semi continuity, their generalizations and the problem of finding a desirable selection are intensively studied in the theory of multifunctions and they play crucial role in many applications. Recall that a multifunction F is upper semi continuous/lower semi continuous (briefly usc/lsc), if $F^+(V) = \{x\colon F(x) \subset V\}/F^-(V) = \{x \in X\colon F(x) \cap V \neq \emptyset\}$ is open for any open set V. In the paper we will be interested in searching a quasi continuous selection. For this goal the upper semi continuity is too strong and the existence of a quasi continuous selection has been studied in a series of papers [1], [9], [10] for more general continuity. Perhaps the most general technic was presented in [1] based on Zorn's lemma. Namely, a compact valued upper Baire continuous multifunction (definition is below) acting from an arbitrary topological space into a regular T_1-space has a quasi continuous selection. Compactness of the values is necessary. A multifunction from $\mathbb{R}$ to $\mathbb{R}$ defined by $F(x) = \{1/x\}$ for $x \neq 0$ and $F(0) = \mathbb{R}$ is upper Baire continuous (even usc) without a quasi continuous selection. The multifunction F above has closed graph which is closely connected to c-upper semi continuity. Namely, F is c-usc (c-upper semi continuous), if $F^{+}(V)$ is open for any open set V with compact complement (see [5], [7], [12]). The dual notion of c-lower semi continuity, briefly c-lsc, means that $F^{-}(V)$ is open for any open set V with compact complement. From the continuity point of view, c-lower semi continuity has very nice behavior. Under reasonable conditions, c-lower semi continuity of F guarantees lower semi continuity of F except for a nowhere dense set [5]. On the other hand, c-upper semi continuity is rather strange. Namely, a c-upper semi continuous multifunction need not be usc/lsc at any point. An example can be found in [5]. The question if a c-upper semi continuous multifunction has a selection (submultifunction) which is quasi continuous (minimal usco) except for a nowhere dense set is the main stimulation for our investigation (see Theorem 4). Moreover, c-upper semi continuity will be replaced by c-u-$\mathcal{E}$-continuity (see Definition 2) which seems to be suitable for finding a selection being quasi continuous except for a nowhere dense set. It is more general than the notion of c-upper semi continuity and closedness of graph, even than the upper Baire continuity, and on the other hand it still leads to reasonable results. The notion of u-$\mathcal{E}$-continuity (formally derived from the upper quasi continuity) is based on a family $\mathcal{E} \subset 2^{X} \setminus \{\emptyset\}$ and the results obtained flexibly depend on a specification of $\mathcal{E}$.

2. Basic definitions and preliminary results

In the sequel X, Y are topological spaces. By $\overline{A}$, A° we denote the closure and the interior of A, respectively. A σ-compact space Y (i.e., $Y = \bigcup_{n=1}^{\infty} C_n$, where C_n are compact) is understood to be Hausdorff. By a multifunction F we understand a subset of the cartesian product $X \times Y$ with the values $\{y \in Y\colon\ [x, y] \in F\} =: F(x)$ (it can be empty valued at some points). For a multifunction F and a set $C \subset Y$, $F \cap C$ denotes the multifunction with the values $F(x) \cap C$. By $\operatorname{Dom}(F)$, we denote the domain of F, i.e., the set of all arguments x at which $F(x)$ is non-empty. A function f is understood as a special multifunction with values $\{f(x)\}$, $x \in \operatorname{Dom}(f)$. For a function f, we will prefer traditional notation of its values as $f(x)$.

A multifunction can be considered as a set-valued mapping from its domain to Y denoted as $F\colon\ A \to Y$, where $A = \operatorname{Dom}(F)$. Then the set $\{[x, y] \in A \times Y\colon\ y \in F(x)\}$ is the graph of F. In the paper, we identify the mapping with its graph.

A multifunction F is bounded on a set A if $F(A) := \bigcup\{F(x)\colon\ x \in A\}$ is a subset of some compact set, and F is locally bounded at x if there is an open set U containing x and a compact set C such that $F(U) \subset C$. If $S \subset F$, then S is called a submultifunction of F. A function f is a selection of a multifunction F, if $f(x) \in F(x)$ for all $x \in \operatorname{Dom}(f) = \operatorname{Dom}(F)$. If $f(x) \in F(x)$ for all $x \in A \subset \operatorname{Dom}(f)$, then f is called a selection of F on a set A. A multifunction F is usco, if $F(x)$ is compact and F is usc at x for all $x \in \mathrm{Dom}\,(F)$.

Any non-empty system $\mathcal{E} \subset 2^X \setminus \{\emptyset\}$ will be called a cluster system. For some special cluster systems we will use a special notation. For example, $\mathcal{O}$, $\mathcal{B}r$ is a cluster system containing all non-empty open sets or all sets being of second category with the Baire property, respectively. For an ideal $\mathbb{I}$ on X, put $\mathcal{E}_{\mathbb{I}} = \{(G \setminus S) \cup T\}$ where $S, T \in \mathbb{I}$ and G is open such that none of its non-empty open subsets is from $\mathbb{I}$.

The next two definitions introduce the notion of an $\mathcal{E}$-cluster point and an upper $\mathcal{E}$-continuity (u-$\mathcal{E}$-continuity), as a basic tool for investigation of the properties of multifunctions. In this form it was studied for the first time in [9], later in [10] and for the functions in [3]. Formally, upper $\mathcal{E}$-continuity (see Definition 2 below) is motivated by the notion of the upper quasi continuity, which is a special case of our approach.

Definition 1. A point $y \in Y$ is an $\mathcal{E}$-cluster point of F at a point x, if for any open sets $V \ni y$ and $U \ni x$ there is a set $E \in \mathcal{E},\ E \subset U$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. The set of all $\mathcal{E}$-cluster points of F at x is denoted by $\mathcal{E}_F(x)$. A multifunction $\mathcal{E}_F$ with the values $\mathcal{E}_F(x)$ is called $\mathcal{E}$-cluster multifunction of F.

Definition 2. A multifunction F is u-$\mathcal{E}$-continuous at $x \in \mathrm{Dom}\,(F)$ (c-u-$\mathcal{E}$-continuous), if for any open sets V, U ($Y \setminus V$ is compact) such that $F(x) \subset V$ and $x \in U$ there is a set $E \in \mathcal{E}$, $E \subset U \cap \mathrm{Dom}\,(F)$ such that $F(e) \subset V$ for any $e \in E$. The global definition is given by the local one at any point of $\mathrm{Dom}\,(F)$. Notation "c-u-$\mathcal{E}$-continuity" is derived from the notion of c-upper semi continuity (see [5], [7]).

Since a function is a special case of a multifunction when upper and lower inverse images coincide, we will say that f is $\mathcal{E}$-continuous, c-$\mathcal{E}$-continuous, respectively. It is evident that if f is $\mathcal{E}$-continuous at x, then $f(x) \in \mathcal{E}_f(x)$. For the system $\mathcal{O}$ we have the notion of upper quasi continuity/c-upper quasi continuity, which is intensively studied, see a survey [13]. A few new characterizations of quasi continuity have been studied in [11]. A u-$\mathcal{B}r$-continuous multifunction is called upper Baire continuous and i his is one of the most general notions of continuity which guarantees the existence of a quasi continuous selection, see [1], [9], [10].

It can happen that some open sets need not contain a set from a given cluster system $\mathcal{E}$. Avoiding such case we can enlarge $\mathcal{E}$ by some reasonable sets, for example by open ones. That is the case of the cluster system $\mathcal{E}_{\mathbb{I}}$ above, which is of our main interest. So we will deal with a cluster system $\mathcal{O} \cup \mathcal{E}_{\mathbb{I}}$ and the continuity introduced in Definition 2 can be considered as the local definition of measurability, i.e., $F^+(V) \cap U$ contains a set of the form $(G \setminus S) \cup T$ (G is open, $S, T \in \mathbb{I}$) whenever $F^+(V) \cap U$ is non-empty. For example, a compact valued multifunction F acting from a Baire space to a metric one has the Baire property if and only if F is u-$\mathcal{E}_{\mathbb{I}}$-continuous except for a set of first category, where $\mathbb{I}$ is the ideal of all sets of first category (see [10]).

Now we give a definition which is a natural generalization of a minimal multifunction ([2], [6], [8]) and in this form has been studied in [11].

Definition 3. A multifunction F is $\mathcal{E}$-minimal at a point x, if $F(x)$ is non-empty and for any open sets U, V such that $U \ni x$ and $V \cap F(x) \neq \emptyset$ there is a set $E \subset U \cap \mathrm{Dom}\,(F)$. The global definition is given by the local one at any point from $\mathrm{Dom}\,(F)$. It is evident that any selection of an $\mathcal{E}$-minimal multifunction is $\mathcal{E}$-continuous.

Lemma 1 (see also [4]). *For any net $\{x_t\}$ converging to x and $y_t \in \mathcal{E}_F(x_t)$, $\mathcal{E}_F(x)$ contains all accumulation points of the net $\{y_t\}$. Consequently, $\mathcal{E}_F$ has a closed graph and closed values.*

Proof. Let y be an accumulation point of $\{y_t\}$. Then for any open sets $V \ni y$ and $U \ni x$ there are frequently given indexes t' such that $x_{t'} \in U$ and $y_{t'} \in V \cap \mathcal{E}_F(x_{t'})$. Hence there is $E \in \mathcal{E}, E \subset U$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. That means $y \in \mathcal{E}_F(x)$. $\square$

Remark 1. Since $\mathcal{E}_F$ has a closed graph, $\mathcal{E}_F^-(K)$ is closed for any compact set K or equivalently, $\mathcal{E}_F^+(G)$ is open for any open G with compact complement in Y. Hence, $\mathcal{E}_F$ is c-upper semi continuous. Consequently, if $\mathcal{E}_F^-(K)$ is dense in an open set G, then $G \subset \mathcal{E}_F^-(K)$, so $\mathcal{E}_F$ is non-empty valued on G.

3. Main results

Lemma 2. *Let Y be Hausdorff.*

(1) *If $F(x)$ is closed, F is c-u-$\mathcal{E}$-continuous at x and S is* usco *at x, then $F \cap S$ is c-u-$\mathcal{E}$-continuous at x provided $F \cap S$ is non-empty on some neighborhood of x.*

(2) *If F is locally bounded and c-u-$\mathcal{E}$-continuous at x, then F is u-$\mathcal{E}$-continuous at x.*

Proof. 1. Let $G \supset F(x) \cap S(x)$ be open with compact complement and let W be open containing x. Then $S(x)$ is disjoint with $F(x) \setminus G$ and since $S(x)$ is compact, there are two disjoint open sets $G_1 \supset S(x)$ and $G_2 \supset F(x) \setminus G$. The complement of $G \cup G_2$ is compact, $G \cup G_2 \supset F(x)$ and by virtue of usc of S and c-u-$\mathcal{E}$-continuity of F, there is an open set $U \subset W$ containing x and there is $E \in \mathcal{E}$, $E \subset U \cap \mathrm{Dom}\,(F)$ such that $F(E) \subset G \cup G_2$ and $S(U) \subset G_1$. Then $F(E) \cap S(E) \subset (G \cup G_2) \cap G_1 \subset (G \cup G_2) \cap (Y \setminus G_2) \subset G$. So $F \cap S$ is c-u-$\mathcal{E}$-continuous.

2. F is a locally bounded multifunction, so there is an open set U containing x and a compact set K such that $F(U) \subset K$. Let $H \supset F(x)$, let H be open and $U_0 \subset U$ open containing x. Since the complement of $(Y \setminus K) \cup H$ is compact, there is a set $E \subset U_0 \cap \mathrm{Dom}\,(F)$ such that $F(E) \subset (Y \setminus K) \cup H$. So $F(E) = F(E) \cap ((Y \setminus K) \cup H) = F(E) \cap H$, which means $F(E) \subset H$. $\square$

Theorem 1. *Let Y be Hausdorff and F compact valued (it can be empty valued at some points) c-u-$\mathcal{E}$-continuous. Then F has a c-$\mathcal{E}$-continuous selection.*

Proof. Let $\mathcal{M}$ be the family of all *c-u-*$\mathcal{E}$-continuous non-empty compact valued submultifunctions of F which is partially ordered by inclusion. It is non-empty, since $F \in \mathcal{M}$. For any linearly ordered subfamily $\mathcal{M}_0$, a multifunction $M_0(x) := \bigcap\{M(x)\colon\ M \in \mathcal{M}_0\}$ is a non-empty compact valued submultifunction of F, and for any open sets $V \supset M_0(x)$, $Y \setminus V$ compact, and U containing x there is $M \in \mathcal{M}_0$ such that $M(x) \subset V$. By the *c-u-*$\mathcal{E}$-continuity of M there is a set $E \in \mathcal{E}$, $E \subset \mathrm{Dom}\,(M) \cap U \cap M^+(V)$, hence for any $e \in E$ we have $M_0(e) \subset M(e) \subset V$. That means M_0 is *c-u-*$\mathcal{E}$-continuous and $\mathcal{M}$ has a minimal element M_m with respect to inclusion. Now we will prove that M_m is $\mathcal{E}$-minimal with respect to co-compact topology on Y given by all open sets with compact complement. If not at $a \in \mathrm{Dom}\,(M_m)$, there is an open set V intersecting $M_m(a)$, $Y \setminus V$ compact and an open set U containing a such that for any $E \subset U \cap \mathrm{Dom}\,(M_m)$ from $\mathcal{E}$ there is a point $e \in E$ such that $M_m(e)$ is not a subset of V. Since M_m is *c-u-*$\mathcal{E}$-continuous, hence for all $u \in U \cap \mathrm{Dom}\,(M_m)$, $M_m(u)$ is not a subset of V. Define a multifunction N as $N(x) := M_m(x)$ if $x \in \mathrm{Dom}\,(M_m) \setminus U$ and $N(x) := M_m(x) \cap (Y \setminus V)$ if $x \in U \cap \mathrm{Dom}\,(M_m)$. Then N is a non-empty compact valued submultifunction of F. We will show that N is *c-u-*$\mathcal{E}$-continuous. If $x \in \mathrm{Dom}\,(M_m) \setminus U$ there is nothing to prove. Let $x \in U \cap \mathrm{Dom}\,(M_m)$, $N(x) \subset W$, let $Y \setminus W$ be compact, $x \in H \subset U$ and H, W be open. Then $M_m(x) \subset V \cup W$ and by the *c-u-*$\mathcal{E}$-continuity of M_m there is a set $E \in \mathcal{E}$, $E \subset H \cap \mathrm{Dom}(M_m)$ such that $M_m(e) \subset V \cup W$ for any $e \in E$. That means $N(e) \subset W$ for any $e \in E$. Hence $N \in \mathcal{M}$ and $N(a)$ is a proper subset of $M_m(a)$, a contradiction with the minimality of M_m. Finally, since M_m is $\mathcal{E}$-minimal with respect to the co-compact topology, any selection of M_m is *c-*$\mathcal{E}$-continuous. $\square$

Remark 2. In a similar way we can prove the next result: If Y is Hausdorff and F is compact valued *u-*$\mathcal{E}$-continuous, then F has an $\mathcal{E}$-continuous selection. For $\mathcal{E} = \mathcal{B}r$ it was proved in [1].

Definition 4. A multifunction is partially $\mathcal{E}$-bounded if for any non-empty open set G there is a set $E \in \mathcal{E}$, $E \subset G$ and a compact set C such that $F(e) \cap C \neq \emptyset$ for any $e \in E$. Hence, a multifunction $F \cap C$ is bounded on E.

Theorem 2. *Let Y be Hausdorff, $\mathcal{E} \subset \mathcal{O} \cup \mathcal{E}_{\mathbb{I}}$ and let F defined on X (i.e., $X = \mathrm{Dom}(F)$) be closed valued and c-u-$\mathcal{E}$-continuous. Then F is partially $\mathcal{E}$-bounded if and only if F has a selection which is both locally bounded and $\mathcal{E}$-continuous except for a nowhere dense set.*

Proof. $\Rightarrow$ We will prove that for any non-empty open set G there is a non-empty open set $G_0 \subset G$ and a compact set C such that $F \cap C$ is non-empty valued on G_0.

By assumption, there are a set $E \in \mathcal{E}$, $E \subset G$ and a compact set C such that

$$(*) \qquad F(e) \cap C \neq \emptyset \quad \text{for any} \quad e \in E.$$

There are two possibilities. Either the set E is open ($E \in \mathcal{O}$) or $E = (G_0 \setminus I) \cup J$ ($E \in \mathcal{E}_{\mathbb{I}}$), where G is open and $I, J \in \mathbb{I}$ and no non-empty open subset of G_0 is from $\mathbb{I}$. First, if E is open, we can put $G_0 = E$. Secondly, if $E = (G_0 \setminus I) \cup J$, we will show that $F(x) \cap C \neq \emptyset$ for any $x \in G_0$. If $F(x) \cap C = \emptyset$ for some $x \in G_0$, then by the *c*-*u*-$\mathcal{E}$-continuity there is $E' \subset G_0 \cap \mathrm{Dom}\,(F)$, $E' \in \mathcal{E}$, such that $F(E') \subset Y \setminus C$. If E' is open, then E' is not from $\mathbb{I}$, so $\emptyset \neq E' \setminus I \subset G_0 \setminus I \subset E$, a contradiction with $(*)$. If $E' = (G' \setminus I') \cup J' \in \mathcal{E}_{\mathbb{I}}$, then E' is not from $\mathbb{I}$ either, and $G' \cap G_0$ is non-empty (otherwise, $E' = E' \cap G_0 \subset (G' \cap G_0) \cup (J' \cap G_0) = J' \cap G_0 \in \mathbb{I}$, a contradiction), so there is a point $a \in G' \cap G_0 \setminus (I' \cup I) \subset E' \cap E$. Hence $F(a) \subset Y \setminus C$ and $F(a) \cap C \neq \emptyset$ (see $(*)$), a contradiction. That means that in both cases $F \cap C$ is a multifunction which is non-empty valued on G_0.

By Lemma 2 (1), $F \cap C$ is non-empty compact valued and *c*-*u*-$\mathcal{E}$-continuous on G_0 and by Theorem 1, $F \cap C$ has a *c*-$\mathcal{E}$-continuous selection f_{G_0} on G_0. Again, f_{G_0} is bounded, so it is $\mathcal{E}$-continuous by Lemma 2 (2).

We have proved for any non-empty open set G there is a non-empty open set $G_0 \subset G$ such that F has a selection that is both bounded and $\mathcal{E}$-continuous on G_0.

Using Zorn's lemma, we can prove the existence of an open set H and a function $f\colon H \to Y$ such that f is both locally bounded and $\mathcal{E}$-continuous and $X \setminus H$ is nowhere dense. So, a function $g\colon X \to Y$ such that $g = f$ on H and $g(x) \in F(x)$ for $x \in X \setminus H$ is a desirable selection.

The converse implication is obvious. $\square$

4. Applications

Global $\mathcal{E}$-continuity on an open set has a very interesting feature. For some cluster systems, global $\mathcal{E}$-continuity of the functions implies quasi continuity. It is the case when Y is regular and $\mathcal{E} \subset \mathcal{O} \cup \mathcal{E}_{\mathbb{I}}$ (see the next theorem or Corollary 1). But in multifunction setting the two notions are different as we can see from the multifunction F defined as $F(x) = \langle 0, 1\rangle$, if x is rational and $F(x) = \{0\}$ otherwise. It is u-$\mathcal{B}r$-continuous but not upper quasi continuous. This is a nice methodological feature of the upper Baire continuity, when a more general continuity of a multifunction guarantees a stronger continuity of a selection, see [1], [9], [10].

Theorem 3. *Suppose that the interior of* $\operatorname{Dom}(\mathcal{E}_f)$ *is non-empty, where f is an arbitrary function. If Y is a regular topological space, then $\mathcal{E}_f$ is $\mathcal{O}$-minimal on the interior of* $\operatorname{Dom}(\mathcal{E}_f)$ *provided* $\mathcal{E} \subset \mathcal{O} \cup \mathcal{E}_{\mathbb{I}}$.

P r o o f. Recall that no $E \in \mathcal{E}_{\mathbb{I}}$ is from $\mathbb{I}$. If not at $x \in (\operatorname{Dom}(\mathcal{E}_f))^{\circ}$, there are the open sets $U \ni x$, V and a set $A \subset U \subset (\operatorname{Dom}(\mathcal{E}_f))^{\circ}$ dense in U such that $\mathcal{E}_f(x) \cap V \neq \emptyset$ and $\mathcal{E}_f(a) \cap (Y \setminus \overline{V}) \neq \emptyset$ for any $a \in A$. Let $y \in \mathcal{E}_f(x) \cap V$. Then there is a set $E \in \mathcal{E}, E \subset U$ such that $f(E) \subset V$.

First, suppose that the set E is of the form $E = (G \setminus S) \cup T \in \mathcal{E}_{\mathbb{I}}$, where G is open and $S, T \in \mathbb{I}$. Then the intersection $G \cap U \neq \emptyset$ (otherwise $E \subset (G \cap U) \cup (T \cap U) = T \cap U \in \mathbb{I}$, a contradiction) so there is a point $a \in A \cap G \cap U$ such that $\mathcal{E}_f(a) \cap (Y \setminus \overline{V}) \neq \emptyset$. Pick up $z \in \mathcal{E}_f(a) \cap (Y \setminus \overline{V})$. Then there is a set $E_0 \in \mathcal{E}$, $E_0 \subset G \cap U$ such that $f(E_0) \subset Y \setminus \overline{V}$ and E_0 is of the form $E_0 = (G_0 \setminus S_0) \cup T_0 \in \mathcal{E}_{\mathbb{I}}$, where G_0 is open and $S_0, T_0 \in \mathbb{I}$ or $E_0 \in \mathcal{O}$. In the first case, the intersection $G \cap U \cap G_0 \neq \emptyset$ (otherwise $E_0 = G \cap U \cap ((G_0 \setminus S_0) \cup T_0) \subset (G \cap U \cap G_0) \cup (G \cap U \cap T_0) = G \cap U \cap T_0 \in \mathbb{I}$, a contradiction), hence $G \cap U \cap G_0 \setminus (S \cup S_0)$ is not from $\mathbb{I}$ and there is a point $e \in G \cap U \cap G_0 \setminus (S \cup S_0) \subset E$, so $f(e) \in V$. On the other hand, $e \in E_0$, so $f(e) \in Y \setminus \overline{V}$, a contradiction. In the second case, when $E_0 \in \mathcal{O}$, $E_0 = E_0 \cap G \cap U$ is a non-empty open subset of G, so E_0 is not from $\mathbb{I}$ and there is a point $e \in E_0 \setminus S$ for which $f(e) \in Y \setminus \overline{V}$. Since $E_0 \subset G$, we have $e \in E$ and $f(e) \in V$, a contradiction.

Secondly suppose, the set E is open subset of U. Then there is a point $a \in A \cap E$ such that $\mathcal{E}_f(a) \cap (Y \setminus \overline{V}) \neq \emptyset$. Pick up $z \in \mathcal{E}_f(a) \cap (Y \setminus \overline{V})$. Then there is a set $E_0 \in \mathcal{E}$, $E_0 \subset E$ such that $f(E_0) \subset Y \setminus \overline{V}$ but $f(E) \subset V$, which is a contradiction. □

Corollary 1. *Let Y be regular and $\mathcal{E} \subset \mathcal{O} \cup \mathcal{E}_{\mathbb{I}}$. If a function f is $\mathcal{E}$-continuous on an open set H, then f is quasi continuous on H.*

P r o o f. Since f is $\mathcal{E}$-continuous, $f(h) \in \mathcal{E}_f(h)$ for any $h \in H$ and by the above theorem, $\mathcal{E}_f$ is $\mathcal{O}$-minimal on H. Then any of its selections is quasi continuous, hence f is also quasi continuous. □

Corollary 2. *Let Y be T_1-regular and $\mathcal{E} \subset \mathcal{O} \cup \mathcal{E}_{\|}$. If a closed valued multifunction F defined on X is c-u-$\mathcal{E}$-continuous and partially $\mathcal{E}$-bounded, then F has a selection which is quasi continuous except for a nowhere dense set.*

Theorem 4. *Let Y be a σ-compact regular space and let F be a closed valued multifunction with* $\mathrm{Dom}\,(F) = X$. *Each of the following conditions ensures the existence of a selection of F which is both locally bounded and quasi continuous except for a nowhere dense set.*

(1) *F is c-upper Baire continuous.*

(2) *X is Baire and F is c-upper quasi continuous.*

*Moreover, if X is Baire and F is c-*usc, *then F has a non-empty valued submultifunction which is both locally bounded and $\mathcal{O}$-minimal* usco *except for a nowhere dense set.*

Proof. (1) Since F is c-upper Baire continuous, X is Baire. We will show $F^+(V)$ has the Baire property for any open set V with compact complement. If not, there is an open set U such that both the sets $X_0 := F^+(V)$ and $X \setminus X_0$ are of second category at any point from U. Let $x \in X_0 \cap U$. By the c-upper Baire continuity there is $E \in Br$, $E \subset U$ such that $F(E) \subset V$. Since E is of second category with the Baire property, $E = (G \setminus I) \cup J$ for some G open and I, J of first category and $G \cap U \neq \emptyset$ (otherwise $E = ((G \setminus I) \cup J)) \cap U = ((G \setminus I) \cap U) \cup (J \cap U) = J \cap U$ is of first category). The set $X \setminus X_0$ is of second category at any point from U, so $((G \cap U \cap (X \setminus X_0)) \setminus I$ is of second category, that means there is a point $e \in ((G \cap U \cap (X \setminus X_0)) \setminus I \subset E$. So $F(e) \not\subset V$, a contradiction with $F(E) \subset V$.

Let $Y = \bigcup\limits_{k \in \mathbb{N}} C_k$, let C_k be compact and G non-empty open. Since $G \subset \bigcup\limits_{k \in \mathbb{N}} F^-(C_k)$, there is m such that $F^-(C_m) = X \setminus F^+(Y \setminus C_m)$ has the Baire property and is of second category, so F is partially $\mathcal{B}r$-bounded. By Theorem 2 and Corollary 1, F has a desirable selection.

(2) Since X is Baire, F is also c-upper Baire continuous and the proof follows from item (1).

Moreover, suppose X is Baire and F is c-usc. Hence F is c-upper Baire continuous and by item (2), F has a selection f which is quasi continuous and locally bounded on an open dense set H. Put $F_0 = \mathcal{B}r_f$. That means $f(h) \in F(h) \cap F_0(h)$ for all $h \in H$. It is clear that F_0 is both locally bounded and with closed graph, so it is usco on H and F_0 is $\mathcal{O}$-minimal on H. Hence for any $x \in H$ there is an open U_0 containing x such that $F_0(U_0) \subset C$, where C is compact. We will show that $F_0(x) \subset F(x)$. If not, there is a point $y \in F_0(x) \setminus F(x)$ and there are two disjoint open sets $V \supset F(x)$ and $W \ni y$ such that $\overline{W} \cap V = \emptyset$ (we use regularity of Y and the closed values of F).

A set $C \cap \overline{W}$ is non-empty compact and disjoint with $F(x)$ and since F is c-usc, there is an open set U containing x, $U \subset U_0$ such that $F(U) \subset Y \setminus (C \cap \overline{W})$. Since F_0 is minimal, there is a non-empty open set $H_0 \subset U$ such that $F_0(H_0) \subset W$. Hence $F_0(H_0) \subset C \cap \overline{W}$. So F and F_0 have disjoint values on H_0, a contradiction with the fact that $f(h) \in F(h) \cap F_0(h)$ for all $h \in H$. Defining G as $G(x) = F_0(x)$ if $x \in H$ and $G(x)$ an arbitrary non-empty subset of $F(x)$ if $x \in X \setminus H$ we obtain a desirable submultifunction of F. □

As we have mentioned, by [5] there is a multifunction F which is c-usc but not usc/lsc at any point. The question is, if there is some reasonable "small" or "big" submultifunction of F. A "small" variant is given in Theorem 4 by proving the existence of a submultifunction which is both $\mathcal{O}$-minimal usco and locally bounded except for a nowhere dense set. The open problem is a "big" variant, namely, to describe a "maximal" usco (usc, lsc) submultifunction of F. More generally, for a c-upper Baire continuous closed (compact) valued multifunction to describe its maximal submultifunction which is lower/upper quasi continuous or usco except for a nowhere dense set.

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Author's address: Milan Matejdes, Department of Mathematics, Faculty of Applied Informatics, Tomas Bata University in Zlín, Nad Stráněmi 4511, 760 05 Zlín, Czech Republic, e-mail: matejdes@fai.utb.cz.