# Continuous-Time and Discrete Multivariable Decoupling Controllers

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*Abstract:* - The paper is focused on a design and implementation of a decoupling multivariable controller. The controller was designed in both discrete and continuous-time versions. The control algorithm is based on polynomial theory and pole – placement. A decoupling compensator is used to suppress interactions between control loops. The controller integrates an on – line identification of an ARX model of a controlled system and a control synthesis on the basis of the identified parameters. The model parameters are recursively estimated using the recursive least squares method.

*Key-Words:* - multivariable control, control algorithms, adaptive control, polynomial methods, pole assignment, recursive identification

## 1 Introduction

Typical technological processes require the simultaneous control of several variables related to one system. Each input may influence all system outputs. The design of a controller for such a system must be quite sophisticated if the system is to be controlled adequately. There are many different methods of controlling MIMO (multi input – multi output) systems. Several of these use decentralized PID controllers [1], others apply single input-single-output (SISO) methods extended to cover multiple inputs [2]. The classical approach to the control of multi-input–multi-output (MIMO) systems is based on the design of a matrix controller to control all system outputs at one time. The basic advantage of this approach is its ability to achieve optimal control performance because the controller can use all the available information about the controlled system. Controllers are based on various approaches and various mathematical models of controlled processes. A standard technique for MIMO control systems uses polynomial methods [3], [4], [5] and is also used in this paper. Controller synthesis is reduced to the solution of linear Diophantine equations [6].

One controller, which enables decoupling control of TITO (two input-two output) systems, is presented. The proposed control algorithm applies a decoupling compensator [7], [8], [9] to suppress undesired interactions between control loops. The controller was realized both in discrete and continuous-time versions. Both versions of the controller were realized both with fixed parameters and as self-tuning controllers [10], [11] with recursive identification of a model of the controlled system. The recursive least squares method is used in the identification part.

The controllers were verified both by simulation and real-time control of a TITO laboratory model of a coupled drives process. The objective laboratory model of the coupled drives apparatus is a nonlinear system with variable parameters. Self-tuning controllers are a possible approach to the control of this kind of system.

## 2 Mathematical Model of the Controlled Process

A general transfer matrix of a two-input–two-output system with significant cross-coupling between the control loops is expressed as (for continuous-time systems $q = s$ as the derivative operator and for discrete systems $q = z^{-1}$ as the delay operator)

$$\boldsymbol{G}(q) = \begin{bmatrix} G_{11}(q) & G_{12}(q) \\ G_{21}(q) & G_{22}(q) \end{bmatrix} \tag{1}$$

$$\boldsymbol{Y}(q) = \boldsymbol{G}(q)\boldsymbol{U}(q) \tag{2}$$

where $\boldsymbol{U}(q)$ and $\boldsymbol{Y}(q)$ are vectors of the manipulated variables) and the controlled variables, respectively.

$$\boldsymbol{Y}(q) = [y_1(q), y_2(q)]^T \; \boldsymbol{U}(q) = [u_1(q), u_2(q)]^T \tag{3}$$

It may be assumed that the transfer matrix can be transcribed to the following form of the matrix fraction:

$$\boldsymbol{G}(q) = \boldsymbol{A}^{-1}(q)\boldsymbol{B}(q) = \boldsymbol{B}_1(q)\boldsymbol{A}_1^{-1}(q) \tag{4}$$

where the polynomial matrices $\boldsymbol{A} \in R_{22}[q]$, $\boldsymbol{B} \in R_{22}[q]$ represent the left coprime factorization of matrix $\boldsymbol{G}(q)$ and the matrices $\boldsymbol{A}_1 \in R_{22}[q]$, $\boldsymbol{B}_1 \in R_{22}[q]$ represent the right coprime factorization of $\boldsymbol{G}(q)$. The further described algorithms are based on a model with polynomials of second order. This model proved to be effective for control of several TITO laboratory processes [12], where controllers based on a model with polynomials of the first order failed. In case of decoupling control using a compensator it is useful to consider matrix $\boldsymbol{A}(q)$ as diagonal. The reason is explained in further section.

### 2.1 Discrete Model

Polynomial matrices of the discrete model are given by following exressions

$$\boldsymbol{A}(z^{-1}) = \begin{bmatrix} 1 + a_1 z^{-1} + a_2 z^{-2} & 0 \\ 0 & 1 + a_3 z^{-1} + a_4 z^{-2} \end{bmatrix} \tag{5}$$

$$\boldsymbol{B}(z^{-1}) = \begin{bmatrix} b_1 z^{-1} + b_2 z^{-2} & b_3 z^{-1} + b_4 z^{-2} \\ b_5 z^{-1} + b_6 z^{-2} & b_7 z^{-1} + b_8 z^{-2} \end{bmatrix} \tag{6}$$

The matrices can be converted to difference equations

$$\begin{aligned} y_1(k) &= -a_1 y_1(k-1) - a_2 y_1(k-2) + \\ &+ b_1 u_1(k-1) + b_2 u_1(k-2) + b_3 u_2(k-1) + b_4 u_2(k-2) \end{aligned} \tag{7}$$

$$\begin{aligned} y_2(k) &= -a_3 y_2(k-1) - a_4 y_2(k-2) + \\ &+ b_5 u_1(k-1) + b_6 u_1(k-2) + b_7 u_2(k-1) + b_8 u_2(k-2) \end{aligned} \tag{8}$$

### 2.2 Continuous-Time Model

Polynomial matrices of the continuous-time model are defined as follows

$$\boldsymbol{A}(s) = \begin{bmatrix} s^2 + a_1 s + a_2 & 0 \\ 0 & s^2 + a_7 s + a_8 \end{bmatrix} \tag{9}$$

$$\boldsymbol{B}(s) = \begin{bmatrix} b_1 s + b_2 & b_3 s + b_4 \\ b_5 s + b_6 & b_7 s + b_8 \end{bmatrix} \tag{10}$$

Differential equations describing dynamical behavior of the system are

$$y_1'' + a_1 y_1' + a_2 y_1 = b_1 u_1' + b_2 u_1 + b_3 u_2' + b_4 u_2 \tag{11}$$

$$y_2'' + a_3 y_2' + a_4 y_2 = b_5 u_1' + b_6 u_1 + b_7 u_2' + b_8 u_2 \tag{12}$$

## 3 Design of Decoupling Controllers

One of possible approaches to control of multivariable systems is the serial insertion of a compensator ahead of the system [7], [8], [9]. The compensator then becomes a part of the controller. The objective, in this case, is to suppress undesirable interactions between the input and output variables so that each input affects only one controlled variable. The block diagram for this kind of system is shown in Fig. 1 ($\boldsymbol{R}$ is a transfer matrix of a controller and $\boldsymbol{C}$ is a decoupling compensator).

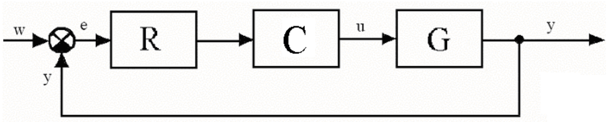


Fig. 1 Closed loop with compensator

The resulting transfer function $\boldsymbol{H}$ (the operator $q$ will be omitted from some operations for the purpose of simplification) is then determined by

$$\boldsymbol{H} = \boldsymbol{G}\boldsymbol{C} = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C} \tag{13}$$

The decoupling conditions are fulfilled when matrix $\boldsymbol{H}$ is diagonal. As it was mentioned above the matrix $\boldsymbol{A}$ is supposed to be diagonal. The reason for this simplification is apparent from equation (13). When matrix $\boldsymbol{A}$ is assumed to be non-diagonal it has to be included into the compensator in order to obtain a diagonal matrix $\boldsymbol{H}$. The order of the controller and consequently complexity of its design would increase. Moreover, a possible incorporation of the matrix $\boldsymbol{A}$ into a compensator is restricted only for compensators where product of the remaining part of the compensator and the matrix $\boldsymbol{B}$ is in a result a diagonal matrix with equal elements in the main diagonal.

### 3.1 Design of Discrete Controller

In case of discrete controller, the matrix $\boldsymbol{B}$ can be written as

$$\begin{aligned} \boldsymbol{B} &= z^{-1} \begin{bmatrix} b_1 + b_2 z^{-1} & b_3 + b_4 z^{-1} \\ b_5 + b_6 z^{-1} & b_7 + b_8 z^{-1} \end{bmatrix} = \\ &= z^{-1} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = z^{-1} \boldsymbol{B}_x \end{aligned} \tag{14}$$

The determinant of the matrix $\boldsymbol{B}_x$ is then defined as

$$\det(\boldsymbol{B}_x) = B_{11} B_{22} - B_{12} B_{21} \tag{15}$$

The compensator is defined as the adjoint matrix $\boldsymbol{B}$

$$\boldsymbol{C} = adj(\boldsymbol{B}_x) \tag{16}$$

The multiplication of the matrix $\boldsymbol{B}_x$ and the adjoint matrix $\boldsymbol{B}_x$ results in a diagonal matrix $\boldsymbol{H}$. The determinants of the matrix $\boldsymbol{B}_x$ represent the diagonal elements.

$$\boldsymbol{H}_1 = z^{-1}\boldsymbol{B}_x adj(\boldsymbol{B}_x) = z^{-1}\begin{bmatrix} det(\boldsymbol{B}_x) & 0 \\ 0 & det(\boldsymbol{B}_x)\end{bmatrix} \quad (17)$$

Generally, the vector of input reference signals $\boldsymbol{W}$ is specified as

$$\boldsymbol{W}(z^{-1}) = \boldsymbol{F}_w^{-1}(z^{-1})\boldsymbol{h}(z^{-1}) \quad (18)$$

Further, the reference signals are considered as step functions. In this case $\boldsymbol{h}$ is a vector of constants and $\boldsymbol{F_w}$ is expressed as

$$\boldsymbol{F}_w(z^{-1}) = \begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1-z^{-1}\end{bmatrix} \quad (19)$$

The controller can be described both by left and right matrix fractions as well as the controlled system

$$\boldsymbol{G}_R(z) = \boldsymbol{P}^{-1}(z^{-1})\boldsymbol{Q}(z^{-1}) = \boldsymbol{Q}_1(z^{-1})\boldsymbol{P}_1^{-1}(z^{-1}) \quad (20)$$

In order to achieve asymptotic tracking of the reference signal, an integrator must be incorporated into the controller. The controller including the integrator can be defined as

$$\boldsymbol{R} = \boldsymbol{F}^{-1}\boldsymbol{Q}_1\boldsymbol{P}_1^{-1} \quad (21)$$

The component $\boldsymbol{F}$ is the integrator. The resulting matrix of the controller can be then defined as follows

$$\boldsymbol{CR} = \boldsymbol{CF}^{-1}\boldsymbol{Q}_1\boldsymbol{P}_1^{-1} \quad (22)$$

It is possible to derive an equation for the system output, which can be modified by matrix operations to the form

$$\boldsymbol{Y} = \boldsymbol{P}_1(\boldsymbol{AFP}_1 + \boldsymbol{HQ}_1)^{-1}\boldsymbol{HQ}_1\boldsymbol{P}_1\boldsymbol{W} \quad (23)$$

The determinant of the matrix in the denominator ($\boldsymbol{AFP}_1$+$\boldsymbol{HQ}_1$) is the characteristic polynomial of the MIMO system. The roots of this polynomial matrix determine the behaviour of the closed loop system. They must be inside the unit circle (of the Gauss complex plane) for the system to be stable. Conditions of BIBO stability can be defined by the following Diophantine matrix equation:

$$\boldsymbol{AFP}_1 + \boldsymbol{HQ}_1 = \boldsymbol{M} \quad (24)$$

where $\boldsymbol{M} \in R_{22}[z^{-1}]$ is a stable diagonal polynomial matrix. If the system has the same number of inputs and outputs, matrix $\boldsymbol{M}$ can be chosen as diagonal, which allows easier computation of the controller parameters. Correct pole placement of the matrix $\boldsymbol{M}$ is very important for good control performance.

$$\boldsymbol{M}(z^{-1}) = \begin{bmatrix} \begin{matrix}1+m_1z^{-1}+ \\ +m_2z^{-2}+m_3z^{-3}+ \\ +m_4z^{-4}+m_5z^{-5}\end{matrix} & 0 \\ 0 & \begin{matrix}1+m_6z^{-1}+ \\ m_7z^{-2}+m_8z^{-3}+ \\ +m_9z^{-4}+m_{10}z^{-5}\end{matrix}\end{bmatrix} \quad (25)$$

The degree of the controller polynomial matrices depends on the internal properness of the closed loop. The structures of matrices $\boldsymbol{P}_1$ and $\boldsymbol{Q}_1$ were chosen so that the number of unknown controller parameters equals the number of algebraic equations resulting from the solution of the Diophantine equation (24) using the method of uncertain coefficients:

$$\boldsymbol{P}_1(z^{-1}) = \begin{bmatrix} 1+p_1z^{-1}+p_2z^{-2} & 0 \\ 0 & 1+p_3z^{-1}+p_4z^{-2}\end{bmatrix} \quad (26)$$

$$\boldsymbol{Q}_1(z^{-1}) = \begin{bmatrix} q_1+q_2z^{-1}+q_3z^{-2} & 0 \\ 0 & q_4+q_5z^{-1}+q_6z^{-2}\end{bmatrix} \quad (27)$$

The solution of the Diophantine equation results in a set of algebraic equations with unknown controller parameters.

For the purpose of a simplification, the det($\boldsymbol{B_x}$($z^{-1}$)) is defined as follows:

$$\begin{aligned} &\det(B_x(z^{-1})) = db_3 + db_2z^{-1} + db_1z^{-2} \\ &\det(B_x(z^{-1})) = (b_1b_7 - b_5b_3) + z^{-1}(b_1b_8 + b_2b_7 - b_5b_4 - b_6b_3) + \\ &+ z^{-2}(b_2b_8 - b_4b_6) \end{aligned} \quad (28)$$

The algebraic equations have the form

$$\begin{bmatrix} 1 & 0 & db_3 & 0 & 0 \\ a_1-1 & 1 & db_2 & db_3 & 0 \\ a_2-a_1 & a_1-1 & db_1 & db_2 & db_3 \\ -a_2 & a_2-a_1 & 0 & db_1 & db_2 \\ 0 & -a_2 & 0 & 0 & db_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} m_1-a_1+1 \\ m_2-a_2+a_1 \\ m_3+a_2 \\ m_4 \\ m_5 \end{bmatrix} \quad (29)$$

$$\begin{bmatrix} 1 & 0 & db_3 & 0 & 0 \\ a_3-1 & 1 & db_2 & db_3 & 0 \\ a_4-a_3 & a_3-1 & db_1 & db_2 & db_3 \\ -a_4 & a_4-a_3 & 0 & db_1 & db_2 \\ 0 & -a_4 & 0 & 0 & db_1 \end{bmatrix} \begin{bmatrix} p_3 \\ p_4 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} m_6-a_3+1 \\ m_7-a_4+a_3 \\ m_8+a_4 \\ m_9 \\ m_{10} \end{bmatrix} \quad (30)$$

The controller parameters are obtained by solving these equations. The parameters are then used for computation of the control law. The control law is defined as:

$$\boldsymbol{FU} = adj(\boldsymbol{B}_x)\boldsymbol{Q}_1\boldsymbol{P}_1^{-1}\boldsymbol{E} \quad (31)$$

where $\boldsymbol{E}$ is a vector of control errors.

### 3.1 Design of Continuous-Time Controller

In case of the continuous-time controller it is not possible to perform a simplifying operation like in case of the discrete controller. The continuous-time compensator is then defined as the adjoint matrix $\boldsymbol{B}$

$$\boldsymbol{C} = adj(\boldsymbol{B}) \tag{32}$$

The matrix $\boldsymbol{H}$ then takes following form

$$\boldsymbol{H} = adj(\boldsymbol{B}) = \begin{bmatrix} det(\boldsymbol{B}) & 0 \\ 0 & det(\boldsymbol{B}) \end{bmatrix} \tag{33}$$

Further procedure is similar like in case of the discrete controller.

$\boldsymbol{F_w}$ in this case is expressed as

$$\boldsymbol{F}_w(s) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \tag{34}$$

The polynomial matrix $\boldsymbol{M}$ takes the form

$$\boldsymbol{M}(s) = \begin{bmatrix} s^5 + m_1 s^4 + & \\ + m_2 s^3 + m_3 s^2 + m_4 s + m_5 & 0 \\ 0 & s^5 + m_6 s^4 + m_7 s^3 + \\ & + m_8 s^2 + m_9 s + m_{10} \end{bmatrix} \tag{35}$$

Polynomial matrices of the continuous-time controller were chosen as follows

$$\boldsymbol{P}_1(s) = \begin{bmatrix} s^2 + p_1 s + p_2 & 0 \\ 0 & s^2 + p_3 s + p_4 \end{bmatrix} \tag{36}$$

$$\boldsymbol{Q}_1(s) = \begin{bmatrix} q_1 s^2 + q_2 s + q_3 & 0 \\ 0 & q_4 s^2 + q_5 s + q_6 \end{bmatrix} \tag{37}$$

For simplification, it was computed the determinant det($\boldsymbol{B}$):

$$\begin{aligned} \det(\boldsymbol{B}) &= (b_1 b_7 - b_3 b_5)s^2 + (b_1 b_8 + b_2 b_7 - b_4 b_5 - b_3 b_6)s + \\ &+ (b_2 b_8 - b_4 b_6) = db_3 s^2 + db_2 s + db_1 \end{aligned} \tag{38}$$

The solution of the Diophantine equation results in a set of 10 algebraic equations with unknown controller parameters. Using matrix notation, the algebraic equations are expressed in the following form.

$$\begin{bmatrix} 1 & 0 & db_3 & 0 & 0 \\ a_1 & 1 & db_2 & db_3 & 0 \\ a_2 & a_1 & db_1 & db_2 & db_3 \\ 0 & a_2 & 0 & db_1 & db_2 \\ 0 & 0 & 0 & 0 & db_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} m_1 - a_1 \\ m_2 - a_2 \\ m_3 \\ m_4 \\ m_5 \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} 1 & 0 & db_3 & 0 & 0 \\ a_3 & 1 & db_2 & db_3 & 0 \\ a_4 & a_3 & db_1 & db_2 & db_3 \\ 0 & a_4 & 0 & db_1 & db_2 \\ 0 & 0 & 0 & 0 & db_1 \end{bmatrix} \begin{bmatrix} p_3 \\ p_4 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} m_6 - a_3 \\ m_7 - a_4 \\ m_8 \\ m_9 \\ m_{10} \end{bmatrix} \tag{40}$$

The control law is defined as:

$$\boldsymbol{FU} = adj(\boldsymbol{B})\boldsymbol{Q}_1 \boldsymbol{P}_1^{-1} \boldsymbol{E} \tag{41}$$

where $\boldsymbol{E}$ is a vector of control errors. This matrix equation can be transcribed to the differential equations of the controller

$$\begin{aligned} & u_1^{(5)} + u_1^{(4)}(p_3 + p_1) + u_1'''(p_4 + p_1 p_3 + p_2) + u_1''(p_1 p_4 + p_2 p_3) + u_1' p_2 p_4 = \\ &= e_1^{(5)} b_7 q_1 + e_1^{(4)}(b_7 q_2 + b_8 q_1 + p_3 b_7 q_1) + e_1'''(b_7 q_3 + b_8 q_2 + p_4 b_7 q_1 + p_3 b_7 q_2 + p_3 b_8 q_1) + \\ &+ e_1''(b_8 q_3 + p_3 b_7 q_3 + p_3 b_8 q_2 + p_4 b_7 q_2 + p_4 b_8 q_1) + e_1'(p_4 b_7 q_3 + p_4 b_8 q_2 + p_3 b_8 q_3) + e_1 p_4 b_8 q_3 - \\ &- e_2^{(5)} b_3 q_4 - e_2^{(4)}(b_3 q_5 + b_4 q_4 + p_1 b_3 q_4) - e_2'''(b_3 q_6 + b_4 q_5 + p_1 b_3 q_5 + p_1 b_4 q_4 + p_3 b_3 q_4) - \\ &- e_2''(b_4 q_6 + p_1 b_3 q_6 + p_1 b_4 q_5 + p_2 b_3 q_5 + p_2 b_4 q_4) - e_2'(p_1 b_4 q_6 + p_2 b_3 q_6 + p_2 b_4 q_5) - e_2 p_2 b_4 q_6 \end{aligned} \tag{42}$$

$$\begin{aligned} & u_2^{(5)} + u_2^{(4)}(p_3 + p_1) + u_2'''(p_4 + p_1 p_3 + p_2) + u_2''(p_1 p_4 + p_2 p_3) + u_2' p_2 p_4 = \\ &= -e_1^{(5)} b_5 q_1 - e_1^{(4)}(b_5 q_2 + b_6 q_1 + p_3 b_5 q_1) - e_1'''(b_5 q_3 + b_6 q_2 + p_4 b_5 q_1 + p_3 b_5 q_2 + p_3 b_6 q_1) - \\ &- e_1''(b_6 q_3 + p_4 b_6 q_1 + p_3 b_5 q_3 + p_4 b_5 q_2 + p_3 b_6 q_2) - e_1'(p_3 b_6 q_3 + p_4 b_5 q_3 + p_4 b_6 q_2) - e_1 p_4 b_6 q_3 + \\ &+ e_2^{(5)} b_1 q_4 + e_2^{(4)}(b_1 q_5 + b_2 q_4 + p_1 b_1 q_4) + e_2'''(b_1 q_6 + b_2 q_5 + p_2 b_1 q_4 + p_1 b_1 q_5 + p_1 b_2 q_4) + \\ &+ e_2''(b_2 q_6 + p_2 b_1 q_5 + p_2 b_2 q_4 + p_1 b_1 q_6 + p_1 b_2 q_5) - e_2'(p_2 b_1 q_6 + p_2 b_2 q_5 + p_1 b_2 q_6) + e_2 p_2 b_2 q_6 \end{aligned} \tag{43}$$

For purposes of simulation, the controller was realized in the Matlab/Simulink environment as an S-function. It was then necessary to obtain its state equations. Further there it is introduced a conversion of the first differential equation (42) to the state equations. The second differential equation (43) was converted similarly. Equation (42) can be itemized as follows

$$\begin{aligned} & u_{1A}^{(5)} + u_{1A}^{(4)}(p_3 + p_1) + u_{1A}'''(p_4 + p_1 p_3 + p_2) + u_{1A}''(p_1 p_4 + p_2 p_3) + u_{1A}' p_2 p_4 = \\ &= e_1^{(5)} b_7 q_1 + e_1^{(4)}(b_7 q_2 + b_8 q_1 + p_3 b_7 q_1) + e_1'''(b_7 q_3 + b_8 q_2 + p_4 b_7 q_1 + p_3 b_7 q_2 + p_3 b_8 q_1) + \\ &+ e_1''(b_8 q_3 + p_3 b_7 q_3 + p_3 b_8 q_2 + p_4 b_7 q_2 + p_4 b_8 q_1) + e_1'(p_4 b_7 q_3 + p_4 b_8 q_2 + p_3 b_8 q_3) + e_1 p_4 b_8 q_3 \end{aligned} \tag{44}$$

$$\begin{aligned} & u_{1B}^{(5)} + u_{1B}^{(4)}(p_3 + p_1) + u_{1B}'''(p_4 + p_1 p_3 + p_2) + u_{1B}''(p_1 p_4 + p_2 p_3) + u_{1B}' p_2 p_4 = \\ &= -e_2^{(5)} b_3 q_4 - e_2^{(4)}(b_3 q_5 + b_4 q_4 + p_1 b_3 q_4) - e_2'''(b_3 q_6 + b_4 q_5 + p_1 b_3 q_5 + p_1 b_4 q_4 + p_3 b_3 q_4) - \\ &- e_2''(b_4 q_6 + p_1 b_3 q_6 + p_1 b_4 q_5 + p_2 b_3 q_5 + p_2 b_4 q_4) - e_2'(p_1 b_4 q_6 + p_2 b_3 q_6 + p_2 b_4 q_5) - e_2 p_2 b_4 q_6 \end{aligned} \tag{45}$$

Equation (44) can be transcribed to the transfer function. It is also possible to establish an auxiliary variable $Z$

$$\begin{aligned} G(s) &= \frac{\begin{aligned} & b_7 q_1 s^5 + (b_7 q_2 + b_8 q_1 + p_3 b_7 q_1)s^4 + (p_4 b_7 q_3 + p_4 b_8 q_2 + p_3 b_8 q_3)s + \\ & + (b_7 q_3 + b_8 q_2 + p_4 b_7 q_1 + p_3 b_7 q_2 + p_3 b_8 q_1)s^3 + p_4 b_8 q_3 + \\ & + (b_8 q_3 + p_3 b_7 q_3 + p_3 b_8 q_2 + p_4 b_7 q_2 + p_4 b_8 q_1)s^2 \end{aligned}}{s^5 + (p_3 + p_1)s^4 + (p_4 + p_1 p_3 + p_2)s^3 + (p_1 p_4 + p_2 p_3)s^2 + p_2 p_4 s} \\ &= \frac{U_{1A}}{E_1} = \frac{U_{1A}}{Z} \frac{Z}{E_1} \end{aligned} \tag{46}$$

By means of the variable $Z$ it is possible to define following equations

$$\begin{aligned} & b_7q_1z^{(5)} + (b_7q_2 + b_8q_1 + p_3b_7q_1)z^{(4)} + (b_7q_3 + b_8q_2 + p_4b_7q_1 + p_3b_7q_2 + p_3b_8q_1)z''' + \\ & + (b_8q_3 + p_3b_7q_3 + p_3b_8q_2 + p_4b_7q_2 + p_4b_8q_1)z'' + (p_4b_7q_3 + p_4b_8q_2 + p_3b_8q_3)z' + \\ & + p_4b_8q_3z = u_{1A} \end{aligned} \tag{47}$$

$$z^{(5)} + (p_3 + p_1)z^{(4)} + (p_4 + p_1p_3 + p_2)z''' + (p_1p_4 + p_2p_3)z'' + p_2p_4z' = e_1 \tag{48}$$

Equation (48) can be converted to a set of differential equations of the first order (state equations). Choice of the state variables is as follows

$$x_1 = z \quad x_2 = z' \quad x_3 = z'' \quad x_4 = z''' \quad x_5 = z^{(4)} \tag{49}$$

And the state equations are

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= x_5 \\ x_5' &= e_1 - (p_3 + p_1)x_5 - (p_4 + p_1p_3 + p_2)x_4 - (p_1p_4 + p_2p_3)x_3 - p_2p_4x_2 \end{aligned} \tag{50}$$

On the basis of the state variables, which are substituted to equation (47), it is possible to derive the first part of the manipulated variable $u_{1A}$

$$\begin{aligned} u_{1A} &= b_7q_1(e_1 - (p_3 + p_1)x_5 + (p_4 + p_1p_3 + p_2)x_4 + (p_1p_4 + p_2p_3)x_3 + p_2p_4x_2) + \\ & + (b_7q_2 + b_8q_1 + p_3b_7q_1)x_5 + (b_7q_3 + b_8q_2 + p_4b_7q_1 + p_3b_7q_2 + p_3b_8q_1)x_4 + \\ & + (b_8q_3 + p_3b_7q_3 + p_3b_8q_2 + p_4b_7q_2 + p_4b_8q_1)x_3 + (p_4b_7q_3 + p_4b_8q_2 + p_3b_8q_3)x_2 + \\ & + p_4b_8q_3x_1 \end{aligned} \tag{51}$$

Similarly it is possible to transcribe equation (45)

$$\begin{aligned} & -b_3q_4z^{(5)} - (b_3q_5 + b_4q_4 + p_1b_3q_4)z^{(4)} - (b_3q_6 + b_4q_5 + p_1b_3q_5 + p_1b_4q_4 + p_3b_3q_4)z''' - \\ & - (b_4q_6 + p_1b_3q_6 + p_1b_4q_5 + p_2b_3q_5 + p_2b_4q_4)z'' - (p_1b_4q_6 + p_2b_3q_6 + p_2b_4q_5)z' - \\ & - p_2b_4q_6z = u_{1B} \end{aligned} \tag{52}$$

$$z^{(5)} + (p_3 + p_1)z^{(4)} + (p_4 + p_1p_3 + p_2)z''' + (p_1p_4 + p_2p_3)z'' + p_2p_4z' = e_2 \tag{53}$$

State variables were chosen similarly as in the previous case

$$x_6 = z \quad x_7 = z' \quad x_8 = z'' \quad x_9 = z''' \quad x_{10} = z^{(4)} \tag{54}$$

The state equations are then as follows

$$\begin{aligned} x_6' &= x_7 \\ x_7' &= x_8 \\ x_8' &= x_9 \\ x_9' &= x_{10} \\ x_{10}' &= e_2 - (p_3 + p_1)x_{10} - (p_4 + p_1p_3 + p_2)x_9 - (p_1p_4 + p_2p_3)x_8 - p_2p_4x_7 \end{aligned} \tag{55}$$

The second part of the manipulated variable $u_{1B}$ can be computed similarly like the part $u_{1A}$ by substitution of the state variables to equation (52)

$$\begin{aligned} u_{1B} &= -b_3q_4(e_2 - (p_3 + p_1)x_{10} + (p_4 + p_1p_3 + p_2)x_9 + (p_1p_4 + p_2p_3)x_8 + p_2p_4x_7) - \\ & - (b_3q_5 + b_4q_4 + p_1b_3q_4)x_{10} - (b_3q_6 + b_4q_5 + p_1b_3q_5 + p_1b_4q_4 + p_3b_3q_4)x_9 - \\ & - (b_4q_6 + p_1b_3q_6 + p_1b_4q_5 + p_2b_3q_5 + p_2b_4q_4)x_8 - (p_1b_4q_6 + p_2b_3q_6 + p_2b_4q_5)x_7 - \\ & - p_2b_4q_6x_6 \end{aligned} \tag{56}$$

The manipulated variable $u_1$ is then defined by the following sum

$$u_1 = u_{1A} + u_{1B} \tag{57}$$

An expression for computation of the manipulated variable $u_2$ is obtained similarly on the basis of differential equation (42).

# 4 Recursive Identification

The controllers were also realized as self-tuning controllers with recursive identification of a model of the controlled system. The recursive least squares method [11] proved to be effective for self-tuning controllers and was used as the basis for our algorithm. For our two-variable example we considered the disintegration of the identification into two independent parts.

It is not possible to measure directly input and output derivatives of a system in case of continuous – time control loop. One of the possible approaches to this problem is establishing of filters and filtered variables to substitute the primary variables. This approach is described in detail in [13]. The filtered variables are then used in the recursive identification procedure.

# 4 Verification of the Controllers

The proposed controllers were verified both by simulation and real-time control of the coupled drives apparatus laboratory model.

## 4.1 Simulation Verification

Verification by simulation was carried out on a range of plants with various dynamics. As a simulation example for the discrete controller it is shown control of a system which represents a linear model of the coupled drives apparatus obtained by the recursive identification for a particular steady state.

$$\boldsymbol{A}(z^{-1}) = \begin{bmatrix} 1 - 0.5827z^{-1} + 0.1745z^{-2} & -0.0220z^{-1} + 0.1797z^{-2} \\ 0.0167z^{-1} - 0.0886z^{-2} & 1 - 0.4564z^{-1} - 0.0830z^{-2} \end{bmatrix} \tag{58}$$

$$\boldsymbol{B}(z^{-1}) = \begin{bmatrix} -0.0035z^{-1} + 0.0955z^{-2} & 0.1484z^{-1} + 0.2197z^{-2} \\ 0.2783z^{-1} + 0.3107z^{-2} & -0.0371z^{-1} - 0.3489z^{-2} \end{bmatrix} \tag{59}$$

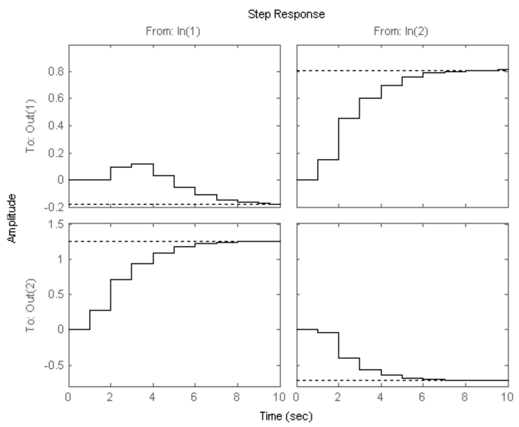


Fig. 2 Step response of the discrete system

A continuous-time model in the form of the matrix fraction obtained by a possible conversion of the discrete model does not need to have the structure on which it is based the computation of the control law. The model obtained by this way would by then unusable.

It is shown control of the following continuous-time system

$$\boldsymbol{A}(s)=\begin{bmatrix} s^2+2s+0{,}7 & 0{,}2s+0{,}4 \\ -0{,}5s-0{,}1 & s^2+2s+0{,}7 \end{bmatrix} \tag{60}$$

$$\boldsymbol{B}(s)=\begin{bmatrix} 0{,}5s+0{,}2 & 0{,}1s+0{,}3 \\ 0{,}5s+0{,}1 & 0{,}3s+0{,}4 \end{bmatrix} \tag{61}$$

Fig. 3 shows the plant‘s step response

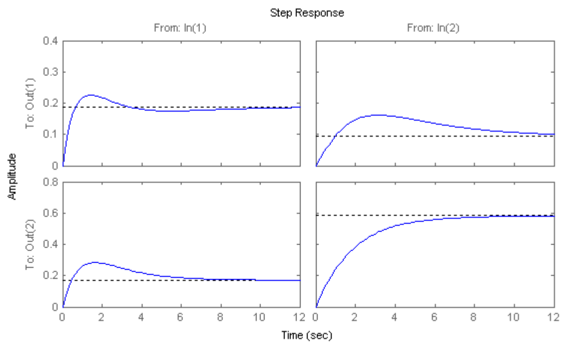


Fig. 3 Step response of the continuous-time system

The controller’s synthesis in both discrete and continuous-time cases is based on the model with diagonal matrix ***A***, which is obtained by recursive identification and which describes the dynamics of the system with full matrix ***A***.

The tuning parameter is the matrix ***M***. A suitable pole-placement (matrix ***M***) was chosen experimentally. At first, a multiple pole was chosen on the real axis. A suitable position of the multiple pole was chosen by experiments and comparison of control results. Then it was searched a suitable combination of various poles in the neighbourhood of the multiple pole.

$$\boldsymbol{M}(s)=\begin{bmatrix} \begin{matrix} s^5+5s^4+10s^3+ \\ +10s^2+5s+1 \end{matrix} & 0 \\ 0 & \begin{matrix} s^5+5s^4+10s^3+ \\ +10s^2+5s+1 \end{matrix} \end{bmatrix} \tag{62}$$

$$\boldsymbol{M}(z^{-1})=\begin{bmatrix} \begin{matrix} 1-0.7z^{-1}+0.01z^{-2}- \\ -0.1z^{-3}-0.05z^{-4}+ \\ +0.0001z^{-5} \end{matrix} & 0 \\ 0 & \begin{matrix} 1-0.7z^{-1}+0.01z^{-2}- \\ -0.1z^{-3}-0.05z^{-4}+ \\ +0.0001z^{-5} \end{matrix} \end{bmatrix} \tag{63}$$

The time responses of the control are shown in Fig. 4-7.

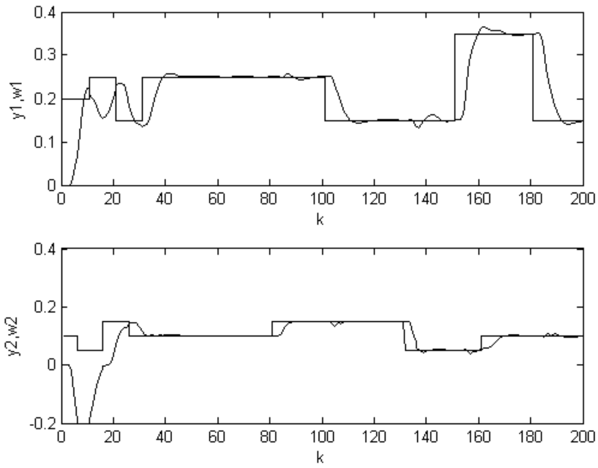


Fig. 4 Adaptive control with discrete controller

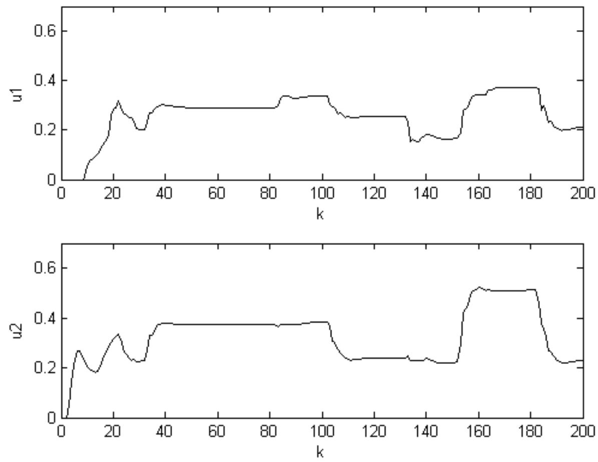


Fig. 5 Adaptive control with discrete controller-manipulated variables

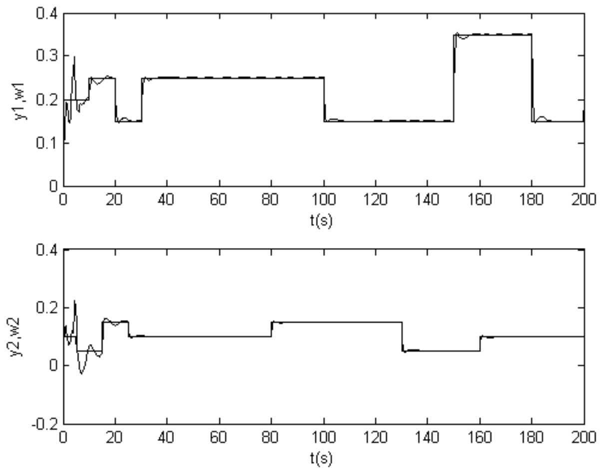


Fig. 6 Adaptive control with continuous-time controller

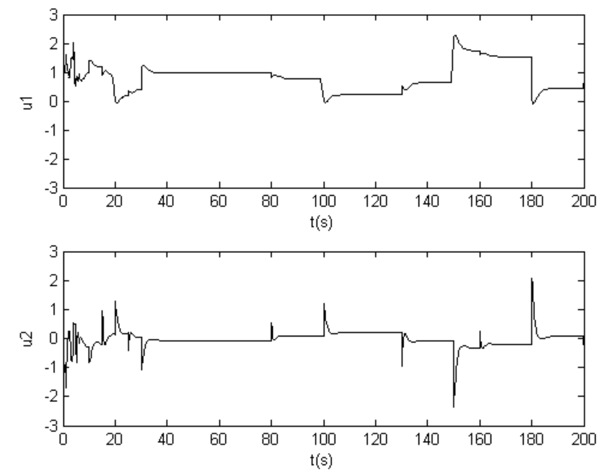


Fig. 7 Adaptive control with continuous-time controller-manipulated variables

From the courses of the variables in Fig.4-7 it is obvious that the basic requirements on control were satisfied. The system was stabilized and the asymptotic tracking of the reference signals was achieved. With regards to decoupling, interactions between the control loops are negligible.

### 4.2 Experimental Verification

The coupled-drives experimental laboratory model was designed to demonstrate simultaneous control of the tension and speed of material in a continuous process. The material passes workstations, where its speed and tension are measured. The material speed and tension must be controlled within the defined limits. The coupled-drives laboratory model represents the standard coupled-drives system, shown in Fig. 8.

The apparatus consists of three pulleys mounted in a vertical panel to form a triangle. The two base pulleys are directly mounted on the shafts of two nominally identical drive motors (motor 1 and motor 2) and the apparatus is controlled by manipulating the drive torques of these motors. The third pulley, the jockey, rotates freely and is mounted on a pivoted arm. The drive motors are coupled by a continuous flexible belt, which also passes over the pivoted arm. The jockey pulley assembly, which simulates a material workstation, is instrumented to allow measurement of the belt speed and tension. The jockey pulley's angular velocity and the belt tension are the system outputs.
The continuous flexible belt couples the actions of motor 1 and motor 2. If a drive voltage to motor 1's drive input is applied, then the speed and the tension in the belt will be changed and motor 2 will be rotated by the drag from motor 1. A similar result is achieved if a drive voltage is applied to motor 2's drive input. Both motors change both outputs. This is the coupling. The manipulated variables are the inputs to the servomotors and the controlled variables are the belt tension and the angular velocity of the jockey pulley. The apparatus can be considered as a two-input–two-output (TITO) system.

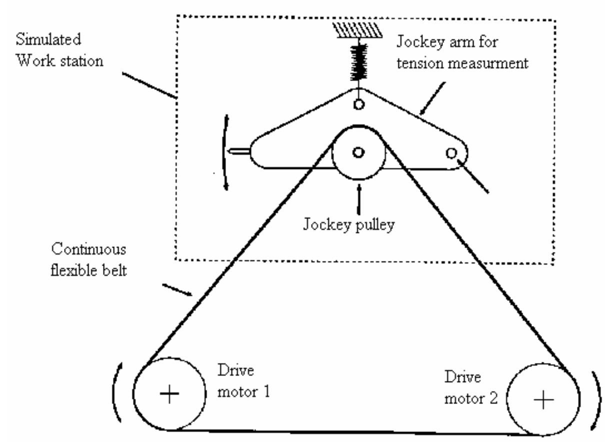


Fig. 8 Principle scheme of coupled drives apparatus

The coupled drives apparatus is a nonlinear system with variable parameters. The nonlinear behaviour is caused by slipping and oscillation of the belt. As the difference in motor speeds increases, the slipping and the oscillations become more apparent. A possible approach to control of nonlinear systems is using of self-tuning controllers. Dynamic behaviour of the system is described in the neighbourhood of a steady state by a linear model (in our case by a model in the form of matrix fraction ). It is an input – output model ("black box model") which does not take into consideration an internal structure of the system. It is a model of the system behaviour and its parameters do not have any particular physical denotation.

The model was connected with a PC equipped with a control and measurement PC card. Matlab and Real Time Toolbox were used to control the system. The best sampling period was found as $T_0 = 0.25$ *s*.

The matrix $\boldsymbol{M}$ was identical as for the simulation example.

Figures 9 and 10 show time responses of the control when the initial parameter estimates were chosen without any a priori information. The reference trajectories contain frequent step changes in the beginning of experiments to activate input and output signals and improve the identification. The manipulated variables $u_1$ and $u_2$ are the inputs to the drive motors 1 and 2. The output $y_1$ is the angular velocity and the output $y_2$ is the tension of the belt.

Subsequent experiment was carried out in such a way that initial parameter estimates were set as the last parameter estimates obtained at the end of the previous experiment. The reference trajectories were chosen to have the same values at the beginning as they had at the end of the previous experiment. This is because the system is nonlinear and the identified parameters were valid only for particular steady states. Time response of this experiment is shown in Fig. 11-12.

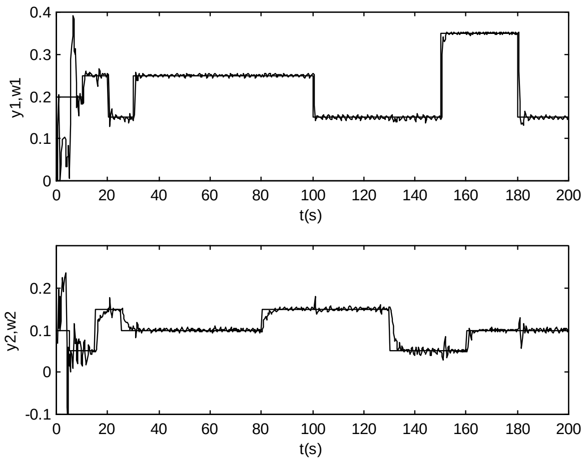


Fig. 9 Discrete control of coupled drives apparatus

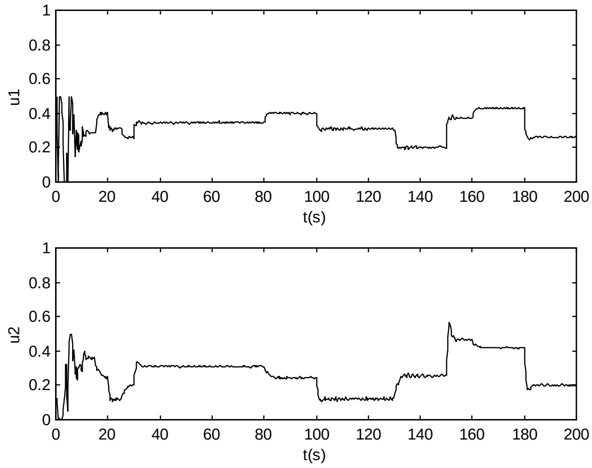


Fig. 10 Discrete control of coupled drives apparatus– manipulated variables

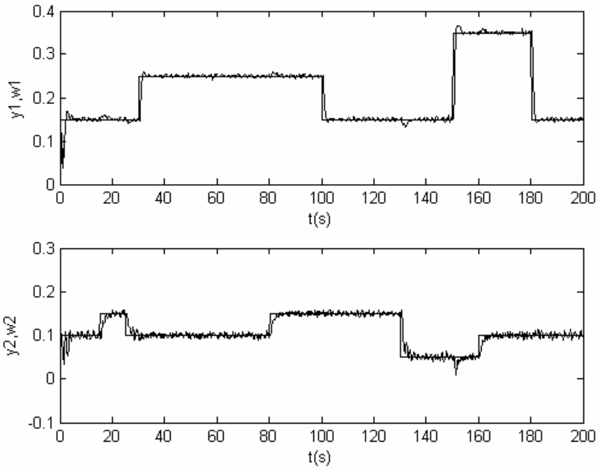


Fig. 11 Discrete control of coupled drives apparatus- experiment with steady parameters

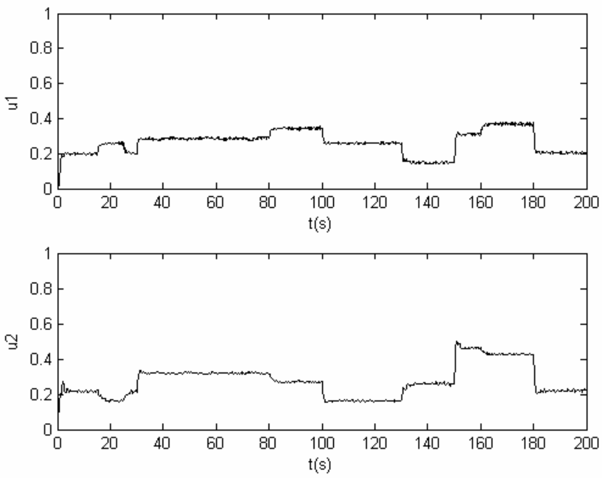


Fig. 12 Discrete control of coupled drives apparatus –manipulated variables-experiment with steady parameters

## 7 Conclusions
Decoupling TITO controller was designed and implemented both in discrete and continuous-time versions. The decoupling compensator was used to suppress interactions between control loops. The adaptive control strategy was also applied.

General principles were elaborated on a specific system with two inputs and two outputs that is often applicable in practice. Control laws based on the specific model was derived in the form of self-contained expressions that is especially useful for practical applications. An advantage of the proposed strategy lies in its simplicity and applicability.

The simulation control tests as well as control tests executed on the laboratory model provide very satisfactory results, despite of the fact, that the non–linear dynamics was described by a linear model. With regards to decoupling, it is clear that the compensator reduces interactions between the control loops.

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