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Teaching Combinatorial Principles Using Relations through the Placemat Method

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Abstract: The presented paper is devoted to an innovative way of teaching mathematics, specifically the subject combinatorics in high schools. This is because combinatorics is closely connected with the beginnings of informatics and several other scientific disciplines such as graph theory and complexity theory. It is important in solving many practical tasks that require the compilation of an object with certain properties, proves the existence or non-existence of some properties, or specifies the number of objects of certain properties. This paper examines the basic combinatorial structures and presents their use and learning using relations through the Placemat method in teaching process. The effectiveness of the presented innovative way of teaching combinatorics was also verified experimentally at a selected high school in the Slovak Republic. Our experiment has confirmed that teaching combinatorics through relationships among talented children in mathematics is more effective than teaching by a standard algorithmic approach.

Keywords: combinations; variations; permutations; the product rule; the sum rule; combinatorics; P and NP problems; relational education; high school; research; talented children; placemat method

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1. Introduction

As a mathematical discipline, combinatorics emerged sometime in the 17th century and was frequently used in connection with gambling and the privileged strata in Western societies. In fact, the history of combinatorics probably goes back to ancient times. In 1858, while visiting the city of Luxor in Egypt, the Scottish antiques dealer Alexander Rhind bought a papyrus from ancient Egypt dated around 1800 BC and inscribed with mathematical symbols. Later, when this papyrus was acquired by the British Museum and the hieroglyphics were translated, it was found that problem no. 79 on the papyrus was a combinatorial one [1,2]. Thomas Kirkman was one of the pioneers of modern combinatorics in the 19th century, and he became famous for the "15 schoolgirls" [3] combinatorial problem. With his original works, Kirkman also contributed to discrete geometry and group theory. Today, combinatorics is an extensive and standalone branch of mathematics, it is important in the development of mathematical thinking and it serves as a basis for solving various probability problems. It mainly deals with tasks related to ordering or sampling. Over the past decade, it has evolved quite rapidly due to the advent and interconnection with computer science. Many problems in other disciplines (e.g., the graph theory and group theory) are solved combinatorically (e.g., the traveling salesman problem, four colors problem, etc.).

Although combinatorics contains no complex mathematical structures, many combinatorial problems suffer due to the sheer complexity of computations, and many real-life problems can only be solved with the use of "computing power". The class of combinatorial problems that can be algorithmically solved in a polynomial time is called P (i.e., practically solvable problems). The wider class of problems includes the problems known as NP, which include combinatorial problems in which we can verify that the solution exists in polynomial time, but an exponential amount of time is needed to find the solution. Thanks to the use of appropriate algorithms, even NP-complete problems can be solved efficiently although they represent the most difficult problems in the NP class. The classification of problems into practically solvable and practically unsolvable is one of the largest discoveries in the theory of complexity [4].

Each combinatorial problem can therefore be characterized by its complexity. In this context, we can understand the complexity of a problem as the effort necessary to solve it. However, this definition of complexity may make us label a problem as more complex just because it is wrongly defined. The various combinatorial problems gave rise to finding the corresponding methods (algorithms), and the big boom of combinatorics was fueled by the use of computers. The measurement of complexity of a given problem by work makes sense for one and the same algorithmic solution. The complexity of a problem can then be judged by the amount of work, which was necessary to calculate the solution using an algorithm.

However, there are various practical combinatorial problems that are not solvable algorithmically, and the absence of an algorithm to address the problem can be proven mathematically—i.e., an algorithmic solution may exist. The mere existence of an algorithm for solving a particular problem does not guarantee that the problem is practically solvable. The practically solvable problems are solved by so-called deterministic algorithms. Each step in the algorithm is unique and it only depends on immediate data. If the solution exists, we know how to arrive at it, or we can arrive at an optimum solution for a larger number of solutions. A deterministic algorithm for practically unsolvable problems requires more time than what is available. Therefore, these problems can only be solved by means of the so-called randomized algorithms that, e.g., can be used to find a partial solution to the problem, or at least a number of solutions according to the specifics of the problem. These algorithms can be visualized as those that randomly select the next step based on the immediate data and according to certain possible strategies. Thus, for the same input problem, this algorithm performs several different calculations, and can even render an incorrect result. Our goal is then to create such conditions for the algorithm that reduce the probability of an incorrect calculation to the minimum.

In practice, randomized algorithms are even used to solve the practically solvable combinatorial problems. There are many problems with a very small probability of an incorrect result from using randomized algorithms, or such a probability is admissible for the calculation. The reason for not using a deterministic algorithm in this case is, for example, the time necessary to complete the calculation compared to a randomized algorithm, which only takes a fraction of the time. For such problems, the relevant deterministic algorithms are practically unusable (for example, primality testing in the context of the number theory where there is no other way to test long prime numbers other than using a randomized algorithm). A proof of strength and meaning behind the use of randomized algorithms goes deep into the foundations of mathematics, and perhaps the most prominent area of application is the various optimization problems, for example in the graph theory or probability theory where it often happens that the result can only be reached by accident. The known combinatorial problems in the graph theory from the NP-complete class are, for example, the decision-making problems such as “is a non-oriented graph k-colorable?”, “Does a (non-)oriented graph contain a (non-)oriented Hamilton circuit?”, ”Does an oriented graph contain a complete subgraph with k vertices?” or “Does a non-oriented graph contain k vertices?” [5,6].

Ideas of combinatorics may appear in many forms, in all common situations such as a game between children, in everyday life and many work areas or in different school subjects. At the beginning, teaching of mathematics is primarily focused on various relationships between objects, but is not specifically concerned with combinatorial problems [7].

Kapur [8] stated that there are several reasons why combinatorics is important and must be taught in school. One of them is that combinatorics does not require a prerequisite calculus so that these topics can be taught early; it does not depend on student mastery of the calculus. Another one is that combinatorics can be used to train students to "count", make estimates, generalize and think systematically. Combinatorics can be applied in many other fields such as programming, physics and engineering, biology, social sciences, management and even a computer application that is widely used and popular. Combinatorics can lead students to understand the strengths and limitations of mathematics. In addition, combinatorics plays an important role in calculation.

Research by Syahputra [9] were focused on determining students' difficulties in the combinatorics problems and creating a mathematical model from the problems given. It can be predicted from the result of the problem and analysis of students' combinatorial thinking that students in general did not understand the problem that was given. Students are not used to the enumeration process in counting. Moreover, almost all students did not make the mathematical model for the problem. Students always used the fastest formula to solve the problem that given.

According to the teaching experience of Spira [10], students have no training whatsoever in combinatorial thinking, because they have been taught that solving combinatorial problems consists mainly of direct computations according to given formulas and using the multiplicative principle. They often then ask, "Does the solution depend on the order of the elements in the set?", "Should I use combinations or variations?", "Can I use the same element from the set in order more times?", etc. In his study he described a few ideas related to the bijection principles which he found useful in actual teaching and which can change students' ways of thinking about and of teaching combinatorics.

Various techniques can be used in the teaching of combinatorics. In the study by Mamona-Downs and Downs [11], the authors implemented one technique that represents well how mental processing of familiar knowledge is not often in a form conducive for application in problem solving. Students might know that a bijection between two finite sets means that the number of elements in each one is equal, but they would not be able to use this fact to create a solving technique to determine a count of elements in a given set, i.e., if they are asked to find the number of elements of a given set, one possible technique that can be used is to find another equivalent set, with which a bijection can be constructed easily.

The book [12], presents a general introduction to enumerative combinatorics that emphasizes bijective methods. Systematic development of the mathematical tools needed to solve enumeration problem and their use for analyzing many combinatorial structures are presented.

2. Mathematical Background

Let us consider the mutually disjunct subsets $X_1, \cdots, X_n$, $n \geq 2$, of a finite set X, $X = X_1 \cup X_2 \cup \cdots \cup X_n$. Then $|X| = |X_1| + |X_2| + \cdots + |X_n|$ and the relation is called a sum rule. The validity of the relation can be shown by mathematical induction according to n. If first $n = 2$ and $X_1 = \{a_1, \cdots, a_k\}$, $X_2 = \{b_1, \cdots, b_m\}$, under the condition $X_1 \cap X_2 = \varnothing$ it implies that $X_1 \cup X_2 = \{a_1, \cdots, a_k, b_1, \cdots, b_m\}$. Then $|X| = |X_1 \cup X_2| = |X_1| + |X_2|$. The second step is analogous. Additionally, for any finite sets, $X_1, \cdots, X_n$, $n \geq 2$, it is true that $|X_1 \times X_2 \times \cdots \times X_n| = |X_1| \cdot |X_2| \cdot \cdots \cdot |X_n|$ and the relation is called a product rule [13]. The relation can also be shown by mathematical induction according to n. For $n = 2$ and the sets $X_1 = \{a_1, \cdots, a_k\}$, and $X_2 = \{b_1, \cdots, b_m\}$, it is true that $X_1 \times X_2 = \{(a_1, b_1), \cdots, (a_1, b_m), (a_2, b_1), \cdots, (a_2, b_m), \cdots, (a_k, b_1), \cdots, (a_k, b_m)\}$. The set therefore contains k m-ths, therefore $|X_1 \times X_2| = k \cdot m = |X_1| \cdot |X_2|$. If the argument already applies to $n > 2$, let us show that it also applies to $n + 1$. We will now determine the number of elements in the set $X_1 \times X_2 \times \cdots \times X_n \times X_{n+1}$. Let $X_{n+1} \neq \varnothing$ and $|X_{n+1}| = s \geq 1$. Then $X_{n+1} = \{c_1, \cdots, c_s\}$. Now let us determine for each $\in \{1, \cdots, s\}$ $Y_i = X_1 \times X_2 \times \cdots \times X_n \times \{c_i\}$. Then $|Y_i| = |X_1 \times X_2 \times \cdots \times X_n|$, and according to the induction assumption, it is true that $|Y_i| = |X_1| \cdot |X_2| \cdot \cdots \cdot |X_n|$. Further, since $X_1 \times X_2 \times \cdots \times X_n \times X_{n+1} = \cup_{i=1}^{s} Y_i$ and the sets $Y_1, \cdots, Y_s$ are mutually disjunct, it follows from the sum rule that $|X_1 \times X_2 \times \cdots \times X_n \times X_{n+1}| = \sum_{i=1}^{s} |Y_i| = |X_1| \cdot |X_2| \cdot \cdots \cdot |X_n|$ [14].

Let A, B be the finite sets, while $|A| = n$ and $|B| = m$. Then $|B^A| = |B|^{|A|} = m^n$ (we denote B^A as the set of all mappings $A \to B$). This relation can be shown by mathematical induction according to n for all-natural numbers m. If first, $n = 0$ then $A = \varnothing$ and the theorem is true because $B^A = \varnothing$. Let the theorem be true for some $n \geq 0$ and $|A| = n + 1$ while $A = \{a_1, \cdots, a_n, a_{n+1}\}$. If $B = \varnothing$ then $B^A = \varnothing$. If $m \geq 1$ and $B = \{b_1, \cdots, b_m\}$, let us define for $k \in \{1, 2, \cdots, m\}$, $Y_k = \{f \in B^A : f(a_{n+1}) = b_k\}$. The sets $Y_k, k = 1, \cdots, m$, are mutually disjunct and $B^A = \cup_{k=1}^{m} Y_k$. The restriction of mappings $f \in Y_k$ on the set $A - \{a_{n+1}\}$ differ by the pairs and they give all the mappings $\{a_1, \cdots, a_n\} \to B$. From the induction assumption it can be implied that $|Y_k| = m^n$. Then $|B^A| = \sum_{k=1}^{m} |Y_k| = m \cdot m^n = m^{n+1} = |B|^{|A|}$. For $A = \{1, 2, \cdots, n\}$ and $|B| = m$, the elements in the set B^A are called variations with repetition of the n-th class consisting of m elements (of set B). For these mappings, the indication $V_n'(m) = m^n$ is used in practice.

For the finite sets, A and B, with $|A| = n$ and $|B| = m$, the number of all injective mappings from A to B is $m \cdot (m-1) \cdot \cdots \cdot (m-n+1) = \prod_{i=0}^{n-1} (m-i)$. If the symbol I_B^A is used to indicate the number of injections $A \to B$ and first let $n = 0$ and $A = \varnothing$, then there is only one injection $A \to B$. There are zero members in the product of $\prod_{i=0}^{n-1} (m-i)$, and such a product is understood as 1. Now let the theorem be true for some $n \geq 0$. Let $|A| = n + 1$ while $A = \{a_1, \cdots, a_n, a_{n+1}\}$. If $B = \varnothing$ then $B^A = \varnothing$. Let $m \geq 1$ and $B = \{b_1, \cdots, b_m\}$ and let us define for $k \in \{1, 2, \cdots, m\}$ the set $Y_k = \{f \in B^A : f(a_{n+1}) = b_k \text{ and } f \text{ is injection}\}$. The sets $Y_k, k = 1, \cdots, m$, are mutually disjunct and each injection $A \to B$ belongs to one of them. Therefore $|Y_1| + |Y_2| + \cdots + |Y_m| = I_B^A$. Let us determine $|Y_k|$ for any k. The restrictions of mappings $f \in Y_k$ to the set $A - \{a_{n+1}\}$ are again the injections $A - \{a_{n+1}\} \to B - \{b_k\}$. Each injection occurs only once between these restrictions, so $|Y_k| = I_{B-\{b_k\}}^{A-\{a_{n+1}\}}$. According to the induction assumption $|Y_k| = \prod_{i=0}^{n-1} (m-i-1) = \prod_{i=1}^{n} (m-i)$. Thence $I_B^A = m \prod_{i=1}^{n} (m-i) = \prod_{i=0}^{n} (m-i)$ [14]. The injections from the set $A = \{1, 2, \cdots, n\}$ into the set B where $|B| = m$ are called variations without repetition (and simply variations or not) of the n-th class consisting of m elements (of set B). For these mappings, the indication $V_n(m)$ is used in practice. The expression $m \cdot (m-1) \cdot \cdots \cdot (m-n+1)$ can be annotated more easily using a factorial $V_n(m) = \frac{m!}{(m-n)!}$ [15]. The variations of the n-th class of elements of the set B are bijection mappings $A \to B$ and their count is $n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1 = n!$. They are called permutations (of set B) and are denoted as $P(n) = n!$ [16].

Let us suppose the finite set A where $|A| = n$. The combinations (without repetition) of the k-th class (or even k-combinations) consisting of n elements are the k-element subsets of the set A. They are denoted as $C_k(n)$ [17] and the count of the k-combinations of the n elements of the set A is $C_k(n) = \binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$ [18]. To show this, let us suppose the set $K = \{0, 1, \cdots, k-1\}$. We will explore the injective mappings of $K \to A$, that is, on the set I_A^K. On this set, we can create the R binary relation $fRg \iff f(\{0, 1, \cdots, k-1\}) = g(\{0, 1, \cdots, k-1\})$. The relation R is then the relation of equivalence, and each class of equivalence C on the set I_A^K is uniquely determined by one k-element subset M, on which the mappings from the set C map the set $\{0, 1, \cdots, k-1\}$. If we replace the codomain A for M in these mappings, we get all permutations of the set M. Therefore $|C| = k!$. Each class of equivalence on I_A^K has $k!$ elements. Therefore $k! \binom{n}{k} = \frac{n!}{(n-k)!} = I_A^K$ and the count of k-element subsets of the set A is $\binom{n}{k} = \frac{I_A^K}{k!} = \frac{n!}{(n-k)!k!}$ [14].

For $A = \{1, 2, \cdots, n\}$ and $|B| = m$ in the set B^A of all variations with the repetition of the n-th class in an m-element set B, we can set relation of equivalence R as follows. Let $f, g \in B^A$. Let us define $fRg \iff \left|f^{-1}(\{x\})\right| = \left|g^{-1}(\{x\})\right|$ for each element $x \in A$. Thus, the two variations with repetition are equivalent if, and only if, the same elements repeat in both of them the same number of times. The combinations with repetition of n-th class made out of m elements in the set B are the classes of equivalence of R on the set B^A. They are denoted as $C'_n(m)$. If B is an m-element set and $n \in \mathbb{N}$, then the count of all the combinations with repetition of the n-th class in the set B is $C'_n(m) = \binom{m+n-1}{n}$. The combinations with repetition of the n-th class in the set B are the elements of decomposition of the set B^A induced by the relation of equivalence $fRg \iff \left|f^{-1}(\{x\})\right| = \left|g^{-1}(\{x\})\right|$. Let $B = \{1, 2, \cdots, m\}$. In every equivalence class of R (combinations with repetition) we select a word, in which the elements of the set B are ordered by size, and we indicate this word with the combination with repetition. Let $c_1 c_2 \cdots c_n$ be a combination with repetition of the n-th class in the set B while $c_1 \le c_2 \le \cdots \le c_n$. Let us assign to this sequence a new sequence $d_1 d_2 \cdots d_n$ by placing $f(c_i) = d_i = c_i + i - 1$, $i = 1, 2, \cdots, n$. It holds that $d_i \in \{1, 2, \cdots, m+n-1\}$ and $d_1 < d_2 < \cdots < d_n$, thus the sequence represents a combination without repetition of the n-th class from the set $\{1, 2, \cdots, m+n-1\}$. The mapping $c_1 c_2 \cdots c_n \to d_1 d_2 \cdots d_n$ is injective. On the other hand, if $e_i \in \{1, 2, \cdots, m+n-1\}$, $e_1 < e_2 < \cdots < e_n$, $i = 1, 2, \cdots, n$ is a combination without repetition of the n-th class. Let us assign the sequence $h_1 h_2 \cdots h_n$ to it so that $h_i = e_i + i - 1$, $i = 1, 2, \cdots, n$. It is true that $h_1 \le h_2 \le \cdots \le h_n$, $h_i \in \{1, 2, \cdots, m\}$. Therefore, $h_1 h_2 \cdots h_n$ is a combination with repetition of the k-th class from the set B, while $f(h_i) = e_i$. The mapping then defines a bijection between the combinations of the n-th class with repetition in the set $\{1, 2, \cdots, m\}$ and combinations without repetition of the n-th class in the set $\{1, 2, \cdots, m+n-1\}$. Therefore, the count of combinations with repetition is $\binom{m+n-1}{n}$ [14].

Let us consider two sets, A and B, while $|A| = n = |B|$. Let set A be decomposed into sets $A_1, A_2, \cdots, A_m$ with a size of $|A_i| = n_i$, $i = 1, 2, \cdots, m$. Let us allow empty sets between the sets A_i and examine the bijections $A \to B$ while two bijections, f, g, will be considered equivalent if for each element of $y \in B$ there is an index $i \in \{1, 2, \cdots, m\}$ so that both elements f^{-1} and g^{-1} belong to the same set A_i. The classes of equivalence of these bijections are denoted as permutations with repetition of n_1 elements of the first type, n_2 elements of the second type, … n_m elements of the m-th type. We mark them $P'_{n_1, \cdots, n_m}(n)$. For the finite sets, A and B, where $|A| = n$, $|B| = m$ and $B = \{b_1, b_2, \cdots, b_m\}$ the number of mappings $f : A \to B$ such that for each element b_i, $i = 1, 2, \cdots, m$, it is true that $\left|f^{-1}(\{b_i\})\right| = n_i$, where n_i are the non-negative integers with the sum $n_1 + n_2 + \cdots + n_m = n$, is equal to $\frac{n!}{n_1! \cdot \cdots \cdot n_m!}$. If we consider $(a_{i_1}, a_{i_2}, \cdots, a_{i_n})$ as a random permutation of set A determined as an ordering, we can define the mapping $A \to B$ so that the first n_1 elements of set A are sent to b_1, the second n_2 elements to b_2, etc. However, the first n_1 elements can be independently permuted and the mapping will not change. This way we can also permute other sets. Thence we get that $n_1!, \cdots, n_m!$ permutations render the same mapping, and each mapping such that $\left|f^{-1}(\{b_i\})\right| = n_i$ for each element $b_i \in B$ is obtained in the same way. Therefore, the number of these mappings is $\frac{n!}{n_1! \cdot \cdots \cdot n_m!}$ [14].

3. Methodology of Research and Discussion

Pedagogical research has been carried out to verify the effectiveness of the innovative approach to teaching the curriculum by a new method—using mappings and relations, with relations and mappings taken separately. The aim of the experiment was to determine whether the selection seminar, where the curriculum was explained by a new method, improved the results of students both in the understanding of the mathematical concept of "combinatorics" and in the improvement of the success of the solution of tasks focused on combinatorial principles. The teaching of combinatorial principles by our proposed method was spread over 2 months with a subsidy of 2 teaching hours per week. Teaching took place in the form of combined education. Students were given study material for homework, which contained basic theoretical knowledge about the latest teaching in the classroom and basic examples. These study materials were made available to students one week before the teaching of the topic in the classroom on the school's website in the e-learning section. Question storming took place in the classroom at the beginning of the meeting, and then the students' questions were answered in the form of a discussion under the supervision of the teacher. The aim of the discussion was for students to gain a conceptual understanding of basic combinatorial concepts. Subsequently, students solved various problems in combinatorics. We used a form of cooperative teaching using the Placemat teaching method [19–21]. For the needs of the experiment, the authors of the study trained a teacher who taught combinatorial principles in a seminar using a new method in EXG.

The experiment was repeatedly carried out among talented children in mathematics at the selected high school in Žilina, Slovak Republic, during four consecutive school years from 2016/2017 to 2019/2020. The experiment was performed in the same way every school year. A total of 104 students participated in the experiment. The students were divided into an experimental and a control group. The experimental group (EXG) consisted of 44 students who preferred the selection seminar. The 60 students who preferred the classical (algorithmic) approach of the teaching of the selected subject formed the control group (CG). During the experiment, both groups of students were taught a selected part of the mathematics curriculum, while the EXG students were also taught the given curriculum by using the new method and the students of the CG were taught the same curriculum only in the standard way.

Before starting the experiment, it was necessary to check the equivalence of both student groups—EXG and CG—in the knowledge of mathematics. We verified the equivalence of both student groups by carrying out a test in combinatorics curriculum. The test (pre-test) included three tasks from the above curriculum, with the results of the pre-test evaluated by marks from A (best) to E (worst). The results achieved by the students of both groups (EXG and CG) in the pre-test are shown in Figure 1.

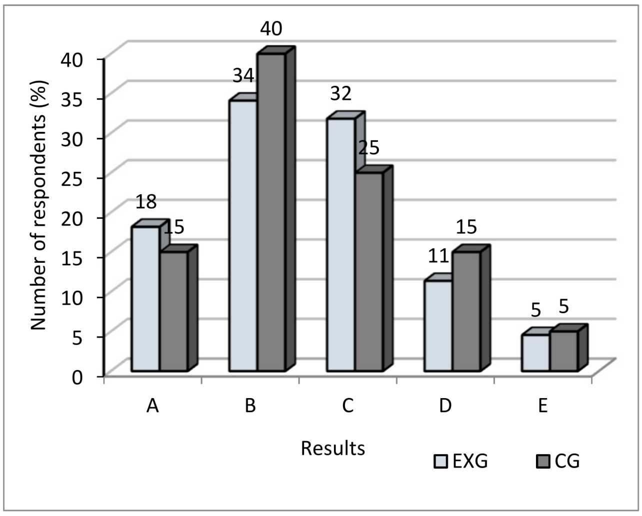

Figure 1. Pre-test results.

Figure 1 shows that the EXG and CG students achieved slightly different results or different grades in the pre-test of the selected mathematics curriculum. We were interested in whether these differences in test results are also statistically significant. The statistical significance of the differences between the EXG and CG students in mathematics pre-test results was verified by the test of qualitative characteristics independence by means of testing statistics Y^2.

A test of independence by means of testing statistics Y^2 is a test by which we can verify independence between two quality characters $\approxeq$ and $\gtrapprox$ whose values are arranged in a contingency table of the type $r \times s$, while the random variable $\approxeq$ acquires values $1, 2, \ldots, r$ ($r \geq 2$) and the random variable $\gtrapprox$ the values $1, 2, \ldots, s$ ($s \geq 2$). Let us denote $p_{ij} = P(\approxeq = i, \gtrapprox = j)$, $p_{i\cdot} = \sum_{j=1}^{s} p_{ij}$ and $p_{\cdot j} = \sum_{i=1}^{r} p_{ij}$, assuming that $p_{ij} > 0$ for $i = 1, 2, \cdots, r$, $j = 1, 2, \cdots, s$ and $\sum_{i=1}^{r}\sum_{j=1}^{s} p_{ij} = 1$ applies. Let us assume that a random selection has been made with the range n from that distribution. Let n_{ij} be empirical rates, i.e., n_{ij} the number of those cases where the selected file pair (i, j) has occurred. The random variables n_{ij} have then become a combined multinomial distribution with the parameter n and probabilities p_{ij}. It is valid that $n = \sum_{i=1}^{r}\sum_{j=1}^{s} n_{ij}$, $n_{i\cdot} = \sum_{j=1}^{s} n_{ij}$, $i = 1, 2, \cdots, r$, $n_{\cdot j} = \sum_{i=1}^{r} n_{ij}$, $j = 1, 2, \cdots, s$, where n_i is a rate of value i of character $\approxeq$, $n_{\cdot j}$ is a rate of value j of character $\gtrapprox$. Numbers $p_{i\cdot}$, $p_{\cdot j}$ are marginal probabilities and values $n_{i\cdot}$ and $n_{\cdot j}$ are marginal rates.

Using testing statistics Y^2, we test the hypothesis H_0: "the characteristics $\approxeq$, $\gtrapprox$ are independent" against an alternative hypothesis H_1: "there is a statistically significant dependence between the characteristics $\approxeq$, $\cdot \gtrapprox$". Testing statistics Y^2 has the form $Y^2 = 2 \cdot \sum_{i=1}^{r}\sum_{j=1}^{s} n_{ij} \cdot \ln \frac{n_{ij}}{e_{ij}}$ [22].

The testing statistics Y^2 have, under the zero-hypothesis validity $H_0 \cdot \chi^2$-, a distribution with $(r-1)(s-1)$ degrees of freedom. We reject the zero hypothesis H_0 on the level of significance α, if the calculated value of the testing statistics Y^2 is greater than the respective critical value χ^2- of the distribution. Yarnold showed that $\chi^2 \approx Y^2$ if the expected rates $e_{ij} > 3$ for all i and j [23].

In our case, the observed characteristics are two quality characteristics, $\approxeq$ and $\gtrapprox$, where $\approxeq$ indicates the results achieved by the EXG students and $\gtrapprox$ indicates the results achieved in the written tests (pre-test) by the CG students.

We have tested the zero hypothesis H_0 of the independence of observed characteristics $\approxeq$ and $\gtrapprox$ on the selected significance level $\alpha = 0.05$, or that the differences between the EXG and CG in the pre-test results are not statistically significant. We have tested the zero hypothesis against the alternative hypothesis H_1 : the differences between the EXG and the CG in the pre-test results are statistically significant.

By the test of independence using testing statistics Y^2, we calculated the value of testing statistics for the contingency table 2×5 $Y^2 = 1.08$. For the selected significance level $\alpha = 0.05$ and the number of degrees of freedom $(5-1)(2-1) = 4$, we search for the critical value of the χ^2- distribution: $\chi^2_{0.05}(4) = 9.49$ (for $\alpha = 0.01$, the critical value is $\chi^2_{0.01}(4) = 13.28$).

Since the calculated value of the testing criterion Y^2 is less than the critical value of 9.49 at the level of significance $\alpha = 0.05$, we cannot reject the tested hypothesis H_0. This means that the differences between the results achieved by the EXG students in the pre-test and the results achieved by the CG students are not statistically significant.

After completing the experiment, both groups (EXG and CG) completed a 90-min written examination (post-test) from the relevant part of mathematics.

Tasks focused on combinatorial calculus were included in the written examination:

Task 1—Show how many times per day the digital clock displays an increasing sequence.

Solution—The time on a digital clock can be encoded with an ordered sextuplet of natural numbers.

$$c = (c_1 c_2 : c_3 c_4 : c_5 c_6)$$

It must be true that $c_1 < c_2 < \cdots < c_6$, while $c_1 < 2$ because if $c_1 = 2$, it would result in $c_5 \geq 6$, which is not possible. Therefore $c_1 \in \{0, 1\}$ and $c_5 \leq 5$.

If $c_1 = 1$ then $c_5 = 5$. If $c_1 = 0$ then $c_5 = 4 \vee c_5 = 5$.

The set H of the sequences we are searching for shall be divided as follows:

$$H_1 = \{c \in H : c_1 = 1\},\ H_{04} = \{c \in H : c_1 = 0, c_5 = 4\},\ H_{05} = \{c \in H : c_1 = 0, c_5 = 5\}$$

In the first set, the sequence has the form $(12:34:5c_6)$, implying $|H_1| = 4$ (because c_6 can only be 6, 7, 8, 9).

In the second set, the sequences have the form $(01:23:4c_6)$ so $|H_{04}| = 5$ (c_6 can only be 5, 6, 7, 8, 9).

The number of elements in the set H_{05} is calculated as follows: for $(c_2,\ c_3,\ c_4)$ the only options are:

$$(1,\ 2,\ 3),\ (1,\ 2,\ 4),\ (1,\ 3,\ 4),\ (2,\ 3,\ 4)$$

For c_6 then 6, 7, 8, 9, (because $c_5 = 5$). Each sequence in H_{05} is characterized by an ordered pair $[(c_2,\ c_3,\ c_4), c_6]$ with a count of $4{\cdot}4 = 16$ according to the product rule.

Overall, according to the product rule, we get $|H| = |H_1| + |H_{04}| + |H_{05}| = 4 + 5 + 16 = 25$.

Task 2—Altogether "k" participants enrolled in the race (the participants start out on the track individually in a set time interval). They include Adam, Peter and George. Show in how many ways the timetable of starts can be arranged so that no two of the above participants start out in succession.

Solution—The total number of all possible timetables is $P(k)$. Let us first determine in how many timetables Adam starts out immediately after Peter. This pair can be seen as a single participant, and the number of such timetables is $P(k-1)$. We arrive at the same count for each of $V(2,3)$ ordered pairs of Adam, Peter and George. This results in the total number of timetables $V(2,3)P(k-1)$, which, however, also include those where all three participants start out in succession. Thus, the number of timetables that must be deducted, is $P(3)P(k-2)$ (each of the ordered triplets $P(3)$ can be taken as one participant). Therefore, the total count is $P(k) - (V(2,3)P(k-1) - P(3)P(k-2))$.

Task 3—Show the number of words of length of k in an n-element alphabet such that (a) there are not two identical successive characters and (b) there are palindromes.

Solution:

(a) The first letter may be chosen from n options, the second from $n-1$ options, the third also from $n-1$ options (we cannot choose the adjacent letter, but the first letter is applicable), etc. Overall, $n(n-1)\cdots(n-1) = n(n-1)^{k-1}$.

(b) The palindrome uniquely identifies the first $(k+1)$ div 2 letters (also known as $\lfloor k+1 \rfloor$—the so-called lower whole part), and the others are thereby determined. If the word has an even number of letters, (k + 1) div 2 is the same as k div 2; if it has an odd number of letters, $(k+1)$ div 2 also includes the middle letter, which is symmetrical with itself.

Overall, we have $n^{\lfloor k+1 \rfloor}$ options.

Task 4—Show the count of all four-digit numbers divisible by nine, which can be written using the digits 0, 1, 2, 5, 7, and the digits in the number may also be repeated.

Solution—The number is divisible by 9 if the sum of its digits is divisible by 9. In our case, only the digit sums 9 and 18 come to mind (no other can be made using the given digits).

Let us first create the instances with the sum of 9 and not take their positions into account just yet:

$$\begin{aligned}
&(\mathrm{a})\ 9 = 7+2+0+0\\
&(\mathrm{b})\ 9 = 7+1+1+0\\
&(\mathrm{c})\ 9 = 5+2+2+0\\
&(\mathrm{d})\ 9 = 5+2+1+1
\end{aligned}$$

We will gradually solve the individual cases.

Case (a), the count of four-digit numbers that can be written using the numbers 7,2,0,0 is $\frac{4!}{2!}$ (these are the permutations of four elements, with one digit repeating twice). However, we must exclude the numbers that have the digit 0 in the first position from the left, i.e., $3! = 6$ numbers (we make a permutation of the remaining three elements). Thus, case (a) has 6 instances ($\frac{4!}{2!} - 3! = 6$).

Case (b) has $\frac{4!}{2!} - \frac{3!}{2!} = 12 - 3 = 9$ instances because digit 1 repeats 2 times in the sequence, and 0 is locked in the first position.

Case (c) has $\frac{4!}{2!} - \frac{3!}{2!} = 12 - 3 = 9$ instances.
Case (d) has $\frac{4!}{2!} = 12$ instances.
Therefore, there are altogether 36 cases of the sum of digit 9.
We will now create the instances with the sum of 18:

$$\text{(a) } 18 = 7 + 7 + 2 + 2$$
$$\text{(b) } 18 = 7 + 5 + 5 + 1$$

In case (a) the number of permutations is $\frac{4!}{2!2!} = 6$ because the first element is repeated 2 times and the second element is also repeated 2 times.

In case (b) the number of permutations is $\frac{4!}{2!} = 12$ because the first element is repeated 2 times.

Therefore, there are altogether 18 cases of the sum of digit 18.

There are altogether 54 numbers divisible by 9, which can be written using the digits 0, 1, 2, 5, 7, and the digits in the number may be repeated.

Task 5—Let us consider domino blocks with every block divided into two halves, and each half marked with one of the values $0, 1, \cdots, n$. None of the two halves can be distinguished as the first or second. Show the probability of how two randomly selected blocks can be attached to each other, i.e., they have the same value at least on one side.

Solution—The block with the values $i, j \in \{0, 1, \cdots, n\}$ can be clearly identified by the set $\{i, j\}$. Since it is possible that $i = j$ (in the game of dominos, we are talking about the so-called doublets), generally for domino block it holds $1 \leq |\{i, j\}| \leq 2$. Therefore, the total number of blocks is $\binom{n+1}{1} + \binom{n+1}{2} = \binom{n+2}{2}$ because the overall count of values in a set $\{0, 1, \cdots, n\}$ is $n+1$. The first element $\binom{n+1}{1}$ represents the number of doublets and the second one the number of blocks with two different values.

The count of all possible selections of two blocks is then:

$$\binom{\binom{n+2}{2}}{2}$$

Now let us show the count of pairs of blocks that share a common value i. In addition to the value i, these blocks also carry two other values j and k ($j \neq k$) while one of these values can be identical to i.

Thus, we get $\binom{n+1}{2}$ pairs of blocks with a common value i because we choose two elements $j,\ k$ from $n+1$ elements. Since the value i can be chosen $(n+1)$-times, we have a total of $(n+1)\binom{n+1}{2}$ pairs of domino blocks that can be attached to each other.

Therefore, the probability of a randomly selected pair with the possibility of mutual attachment is

$$\frac{(n+1)\binom{n+1}{2}}{\binom{\binom{n+2}{2}}{2}} = \frac{2(n+1)\binom{n+1}{2}}{\binom{n+2}{2}\left(\binom{n+2}{2} - 1\right)}$$

Furthermore, in this case, the written (post-test) examination was evaluated by marks from A (best) to E (worst). The results of the post-test in both student groups were analyzed and compared, in each of the school years mentioned above. The results achieved by the students of both groups (EXG and CG) in the post-test in each year are shown in Figures 2–5.

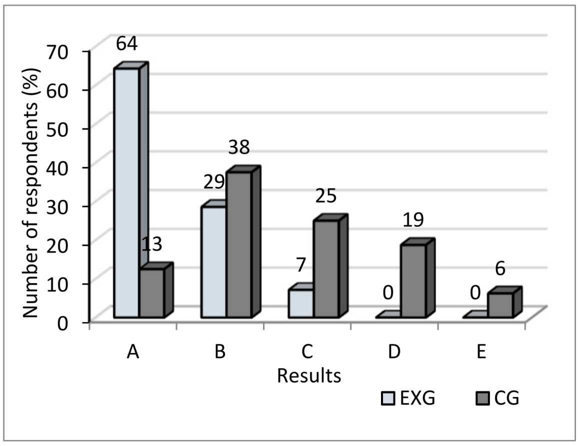

Figure 2. Post-test results in 2016/2017.

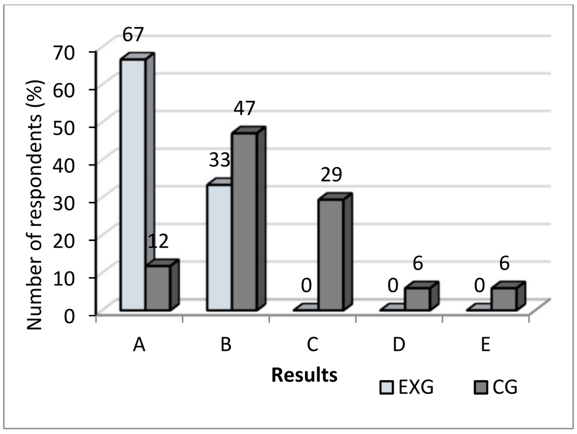

Figure 3. Post-test results in 2017/2018.

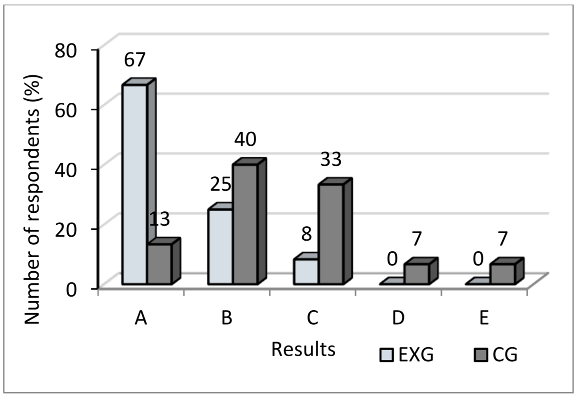

Figure 4. Post-test results in 2018/2019.

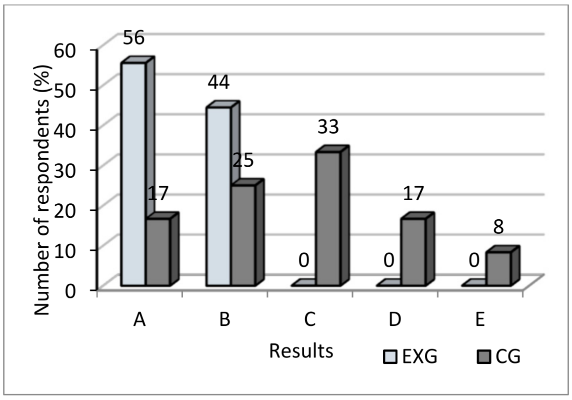

Figure 5. Post-test results in 2019/2020.

From Figures 2–5, we can see that the EXG and CG students achieved different results or different marks in the written examination (post-test) from the selected part of mathematics in each of the years observed. We were interested in whether these differences in the results of written examinations were also statistically significant. The importance of statistical differences between the EXG and CG students in the results of the mathematical post-test results was in this case also verified by the tests of quality characteristics independence in each school year using testing statistics Y^2. In our case, there were also two qualitative characteristics in the school year 2016/17, $\approxeq$ and $\succapprox$, where $\approxeq$ indicates the results that the EXG students achieved in the post-test from selected mathematics and $\succapprox$ indicates the results achieved in the post-test by the CG students. Furthermore, in this case, we tested the zero hypothesis H_0 of independence of observed characteristics, $\approxeq$ and $\succapprox$, on the selected level of significance $\alpha = 0.05$. Or we tested H_0, where the differences between EXG and CG in post-test results are not statistically significant, against the alternative hypothesis H_1, where the differences between the EXG and the CG post-test results are statistically significant.

By the test of independence using testing statistics Y^2, for the contingency table 2×5 we calculated the value of testing statistics $Y^2 = 12.56$. For the selected significance level $\alpha = 0.05$ and for the number of degrees of freedom $(5-1)(2-1) = 4$, we search for the critical value of the χ^2- distribution: $\chi^2_{0.05}(4) = 9.49$ (for $\alpha = 0.01$, the critical value is $\chi^2_{0.01}(4) = 13.28$).

As the calculated test criterion value Y^2 exceeds the critical value of 9.49, we reject the test hypothesis $\alpha = 0.05$ on the significance level H_0 in favour of the alternative hypothesis H_1. This means that the differences between the results achieved by the EXG students in the written examination on mathematics and the results achieved by the CG students in the shown test are statistically significant.

Based on an analysis of the results of the written examination shown in Figure 2, we can see that while only 13% of students in the CG were given an A rating from a written examination, 64% of students in the EXG were rated. In contrast, D and E ratings by the EXG were not achieved by a single student, but by the CG, up to 25% of students achieved this rating.

We also followed an analogous process in verifying the statistical significance of the differences between the EXG and CG in the post-test results from the selected part of mathematics in other school years from 2017/2018 to 2019/2020. By means of testing statistics Y^2, we calculated contingency tables 2×5 and testing statistics values Y^2 that are clearly entered in Table 1.

Table 1. Calculated values of testing statistics Y^2.

School Year	$\boldsymbol{Y^2}$
2017/2018	11.65
2018/2019	10.22
2019/2020	10.74

As the calculated test criteria $\boldsymbol{Y^2}$ in all three school years exceed the critical value of 9.49, $\alpha = 0.05$ in all three cases, we reject the tested hypothesis H_0 in favour of an alternative hypothesis H_1. This means that the differences between the results achieved by the EXG students in the written examination from a selected part of the mathematics curriculum and the results achieved by the CG students in the shown test are statistically significant. In the school year 2016/2017, as well as in other school years, we can see from the results shown in Figures 2–5 that, whereas in the EXG, more than 80% of students achieved an A or B evaluation in the written examination on selected actuarial mathematics, less than 60% of students achieved this rating in the CG. On the contrary, the results of the above tests in the CG were much more frequently evaluated as C–E marks (50%, 41%, 47%, and 58%) each year.

Based on the results of the experiment, we can conclude that by involving a selection seminar (combined with a classic learning approach) a higher level of knowledge was achieved, a better understanding of various principles and algorithms, and, thus, students better mastered the issue. Therefore, in further research, it is effective and necessary to pay attention to the creation of new methods of teaching mathematics, so that innovative teaching methods can be used to a greater extent in the teaching of mathematics.

4. Conclusions

Education is currently being moved to the online sphere, as a result of which, blended learning is gaining in popularity and importance. Changing the context of education also requires the creation of new methodological procedures that will support students, especially in their education outside the classroom, where the presence of a teacher is absent. Additionally, the activities within the classroom need to be innovated so that they are more focused on the application of already acquired theoretical knowledge. Furthermore, on the basis of our research, it has been shown that if students are presented with new knowledge in a suitable way, they also have a better mastery of the curriculum.

Based on a graphical representation of the results of both tests, we can conclude that the probability of obtaining a better grade has increased from attending the selection seminar, where the teaching of combinatorics took place in the manner described above. This has shown that the proposed new way of teaching combinatorics through relations is effective.

It should be noted that numerous real-world problems can be looked upon through combinatorics, and that each combinatorial problem is characterized by complexity as one of its essential features. From this point of view, we support the effort to study and develop the complexity theory, which is based on different mathematical principles. We put emphasis on the formulation of selected problem solutions, which are the basis for understanding the very possibilities in solving combinatorial problems. The theoretical basis of this paper characterizes and describes the basic combinatorial structures, but given its rather narrow scope, a sufficient coverage of such a broad topic as combinatorics, theory of complexity and graph theory can only be found in the relevant literature where the individual sections mentioned in this paper are explained in greater detail.

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