Convexity and level sets for interval-valued fuzzy sets

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Abstract

Convexity is a deeply studied concept since it is very useful in many fields of mathematics, like optimization. When we deal with imprecision, the convexity is required as well and some important applications can be found fuzzy optimization, in particular convexity of fuzzy sets. In this paper we have extended the notion of convexity for interval-valued fuzzy sets in order to be able to cover some wider area of imprecision. We show some of its interesting properties, and study the preservation under the intersection and the cutworthy property. Finally, we applied convexity to decision-making problems.

Keywords Interval-valued fuzzy set · Epistemic interpretation · Intersection · Level set · Convexity · Decision-making

1 Introduction

In a decision-making procedure there are at least three important components to take into account, (1) a set of alternatives, (2) a set of constraints on the option within several alternatives, and (3) a utility function that maps the profit or loss emerging

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from the preference of that alternative with each decision. In many real situations, it is extremely difficult to stipulate precisely the objective function and the constraints. Moreover, sometimes certain vagueness cannot be avoided. In order to deal with imprecision, fuzzy sets can be a very useful tool.

The necessity to deal with imprecision in real world problems has been a long-term research challenge that has originated different extensions of fuzzy sets, interval-valued fuzzy sets (IVFS) being one of them. IVFS can be useful to deal with situations where the classical fuzzy tools are not so efficient as, for instance, when there is not an objective procedure to select the crisp membership degrees. This extension has attracted very quickly the attention of many researchers, since they could see the high potential of them for different applications.

On the other hand, convexity is a fundamental mathematical concept that has helped many researchers to analyze numerous problems. It has become powerful for its applications in diverse areas like optimization (Liberti 2004), image processing (Tofighi et al. 2015), among several others.

In the literature, fuzzy convexity has been deeply studied because most of the real-life situations include approximate information. The need for managing imprecision in real-world problems has been a drawn-out exploration challenge that has originated several extensions of fuzzy sets. Thus, many authors studied different types of convexity of fuzzy sets and its extensions (see, for instance, Ammar and Metz 1992; Díaz et al. 2017; Huidobro et al. 2021; Syau and Lee 2006; Zhang et al. 2016).

Based on their utility, several concepts, tools and trends related to IVFS can be studied. In particular, we are interested in convexity. The main aim of this paper is to find a proper definition of convexity for IVFS. In order to consider the suitability of this definition, we require two properties: it has to be compatible with the intersection and fulfil the cutworthy approach. This means that some properties of fuzzy sets should be reflected in corresponding properties of their cuts. In particular, we look for a definition of convexity that preserves convexity under intersections. This requires a detailed study of the intersection itself. Later we study the union to relate IVFS and the notion of level sets. Then we recover an IVFS knowing only its level sets, which is the topic of the Decomposition Theorem. In the literature there is a lot of papers related to these topics (see, e.g., Huidobro et al. 2020; Yuan and Li 2009). Finally we prove the properties of this proposal for convexity.

This paper is organized as follows. In Sect. 2 basic concepts and notations are introduced. Section 3 is devoted to study the intersection, the union of IVFS and we propose a proper definition of a level set for IVFS and a possible adaptation of the Decomposition Theorem. In Sect. 4, convexity of IVFS is presented and related to previous sections. In Sect. 6, an application of IVFS is presented in a decision-making problem. Some conclusions are drawn in Sect. 6.

2 Basic concepts

Let X denote the universe of discourse. An IVFS on X is a mapping $A : X \to L([0, 1])$ such that $A(x) = [\underline{A}(x), \overline{A}(x)]$, where $L([0, 1])$ denotes the family of closed intervals included in the unit interval $[0, 1]$. Thus, an IVFS A is totally characterized by two
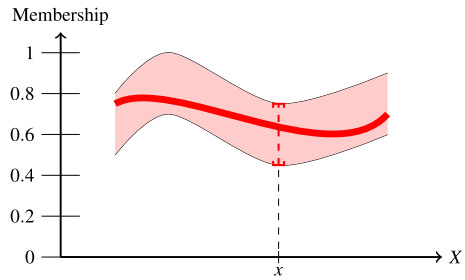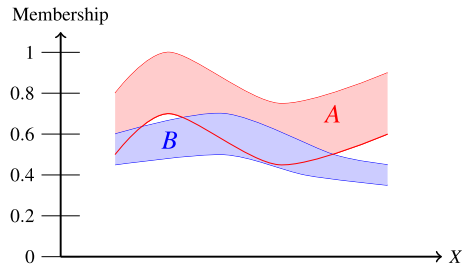
Fig. 1 Epistemic interpretation

Membership

Fig. 2 Is B "included" in A?

Membership

mappings, $\underline{A}$ and $\overline{A}$, from X into [0, 1] such that $\underline{A}(x) \leq \overline{A}(x), \forall x \in X$. These maps represent the lower and upper bound of the corresponding intervals. Let us notice that if $\underline{A}(x) = \overline{A}(x), \forall x \in X$, then A is a classical fuzzy set. The collection of all the IVFS in X is denoted by $IVFS(X)$ and the subset formed by all the fuzzy sets in X is denoted by $FS(X)$.

For IVFS we can consider the epistemic or the ontic interpretation. In our study, the first one will be the chosen one. Thus, we assume that there is one actual, real-valued membership degree of an element inside the membership interval of possible membership degrees, as it is shown in Fig. 1.

A real-life situation when IVFS can be useful might be an expert statement about the truth degree of a statement. In a situation of a complete information the expert should pick a particular value from the unit interval. However, if the information is not complete, the expert may provide just an interval for his estimation.

Since we want our proposal of a convex IVFS to be preserved under intersection, we have to start by defining a concept of the intersection of two IVFS. In order to do that in a coherent way, we need to define the inclusion between two IVFS. Let us consider the following two IVFS at Fig. 2.

It seems natural that we have to compare intervals in order to decide if B is included in A or not.

Apart from that, we would like to find a definition for the convexity such that it has the cutworthy property. In order to study the properties of the level sets of an IVFS we need to also define and study the union, which will also be depending on the considered order between intervals. We start by the study of different ways to order real intervals, since they will be essential to define the inclusion in $IVFS(X)$ and therefore, the union and intersection of IVFS.
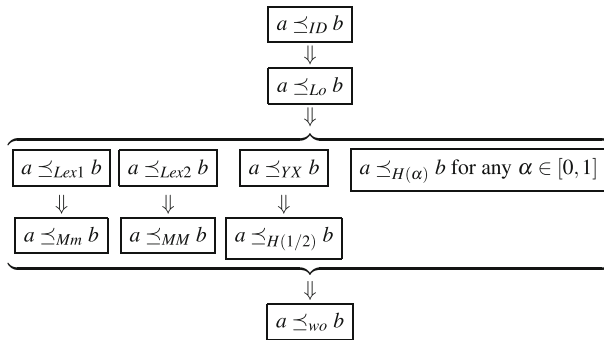
$$a \preceq_{ID} b$$

$$\Downarrow$$

$$a \preceq_{Lo} b$$

$$\Downarrow$$

$a \preceq_{Lex1} b$	$a \preceq_{Lex2} b$	$a \preceq_{YX} b$	$a \preceq_{H(\alpha)} b$ for any $\alpha \in [0,1]$
$\Downarrow$	$\Downarrow$	$\Downarrow$	
$a \preceq_{Mm} b$	$a \preceq_{MM} b$	$a \preceq_{H(1/2)} b$	

$$\Downarrow$$

$$a \preceq_{wo} b$$

Fig. 3 Relations between orders

2.1 Orders

There are several ways to compare intervals and here are the most common ones presented in Huidobro et al. (2020). If $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$ are two intervals in $L([0,1])$, we say that a is lower than or equal to b if:

- Interval dominance order: $a \preceq_{ID} b$ if $\overline{a} \leq \underline{b}$.
- Lattice order: $a \preceq_{Lo} b$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, which is induced by the usual partial order in $\mathbb{R}^2$.
- Lexicographical order type 1: $a \preceq_{Lex1} b$ if $\underline{a} < \underline{b}$ or $\underline{a} = \underline{b}$ and $\overline{a} \leq \overline{b}$.
- Lexicographical order type 2: $a \preceq_{Lex2} b$ if $\overline{a} < \overline{b}$ or $\overline{a} = \overline{b}$ and $\underline{a} \leq \underline{b}$.
- The Xu and Yager order: $a \preceq_{YX} b$ if $\underline{a} + \overline{a} < \underline{b} + \overline{b}$ or $\underline{a} + \overline{a} = \underline{b} + \overline{b}$ and $\overline{a} - \underline{a} \leq \overline{b} - \underline{b}$.
- Maximin order: $a \preceq_{Mm} b$ if $\underline{a} \leq \underline{b}$.
- Maximax order: $a \preceq_{MM} b$ if $\overline{a} \leq \overline{b}$.
- Hurwicz order: $a \preceq_{H(\alpha)} b$ if $\alpha \cdot \underline{a} + (1 - \alpha) \cdot \overline{a} \leq \alpha \cdot \underline{b} + (1 - \alpha) \cdot \overline{b}$ where $\alpha \in [0, 1]$.
- Weak order: $a \preceq_{wo} b$ if $\underline{a} \leq \overline{b}$.

Some of these orders are interrelated. It is well-known that if an interval a is lower than or equal to b w.r.t. the order ID, then a is lower than or equal to b also w.r.t. the lattice order. In Fig. 3 there are summarized all these implications and some other similar ones.

At a first sight, the reader could think that these expressions are truly orders, but this is not true. As we can see in Table 1, some of these ways to compare intervals are not orders as they do not fulfill the order relation requirements (reflexive, antisymmetric and transitive). However, we will refer to all of them as orders. In Table 1, we also claim whether they are total orders or not.

After this simple study, we can affirm that lexicographical orders type 1 and 2, and the Xu and Yager order are the only ones that are total orders.

In Table 1 we have studies some typical examples of ranking methods for interval orders. However, it is clear that there are many others, which are very important and useful in some context as, for example, Liu (2009), Xia and Chen (2015) or Liu et al. (2018). However, we are interested on total orders in $L([0,1])$. Thus, in this work we

Table 1 Properties of the different relations

	Reflexive	Antisymmetric	Transitive	Preorder	Order	Total order
ID	✗	✓	✓	✗	✗	✗
Lo	✓	✓	✓	✓	✓	✗
Lex_1	✓	✓	✓	✓	✓	✓
Lex_2	✓	✓	✓	✓	✓	✓
YX	✓	✓	✓	✓	✓	✓
Mm	✓	✗	✓	✓	✗	✗
MM	✓	✗	✓	✓	✗	✗
$H(\alpha)$	✓	✗	✓	✓	✗	✗
wo	✓	✗	✗	✗	✗	✗

are considering the general family of admissible orders, whose definition it is reviewed below.

Definition 1 (Bustince et al. 2013) An admissible order on $L([0, 1])$ is a total order $\preceq_{to}$ that refines the lattice order; that is, for every $a, b \in L([0, 1])$, if $a \preceq_{Lo} b$ then $a \preceq_{to} b$.

An interesting property of admissible orders is that they can be built using aggregation functions Bustince et al. (2013).

Definition 2 (Beliakov et al. 2016; Mesiar and Komorníková 2011) Let $\mathscr{A}$: $\bigcup_{i=1}^{n}[0, 1]^{i} \to [0, 1]$ such that

- $\mathscr{A}(0, 0, \ldots, 0) = 0$, $\mathscr{A}(1, 1, \ldots, 1) = 1$,
- $\mathscr{A}(x) = x$ for all $x \in [0, 1]$,
- $\mathscr{A}$ is monotone in each variable,

then $\mathscr{A}$ is an aggregation function.

There is a natural bijection between $L([0, 1])$ and $K([0, 1]) = \{(u, v) \in [0, 1]^2 \mid u \leq v\}$ that associates an interval $[\underline{a}, \overline{a}]$ to the point in $\mathbb{R}^2$ created by its endpoints, that is, $(\underline{a}, \overline{a})$ (see Bustince et al. 2013). Thus, we can use aggregation functions to sum up the information stated by an interval. Based on this idea, Bustince et al. construct the following method to make admissible orders.

Proposition 1 (Bustince et al. 2013) Let $\mathscr{A}, \mathscr{B} : [0, 1]^2 \to [0, 1]$ be continuous aggregation functions, such that for all $(u, v), (u', v') \in K([0, 1])$, the equalities $\mathscr{A}(u, v) = \mathscr{A}(u', v')$ and $\mathscr{B}(u, v) = \mathscr{B}(u', v')$ can only hold if $(u, v) = (u', v')$. Define the relation $\preceq_{\mathscr{A},\mathscr{B}}$ on $L([0, 1])$ by $a \preceq_{\mathscr{A},\mathscr{B}} b$ if and only if

$$\mathscr{A}(\underline{a}, \overline{a}) < \mathscr{A}(\underline{b}, \overline{b})$$

or

$$\mathscr{A}(\underline{a}, \overline{a}) = \mathscr{A}(\underline{b}, \overline{b}) \text{ and } \mathscr{B}(\underline{a}, \overline{a}) \leq \mathscr{B}(\underline{b}, \overline{b}).$$

Then $\preceq_{\mathscr{A},\mathscr{B}}$ is an admissible order on $L([0, 1])$.

A possible procedure of building admissible orders on $L([0, 1])$ is defining them using the weighted mean (see Bustince et al. 2013):

$$K_\alpha(u, v) = (1 - \alpha) \cdot u + \alpha \cdot v, \quad \text{where} \quad \alpha \in [0, 1].$$

The α-quantile of a probability distribution uniformly distributed over the interval $[u, v]$ can be represented by this mapping. If $\alpha \neq \beta$, we can apply Proposition 1 to the aggregation functions K_α and K_β in order to obtain the admissible order $\preceq_{K_\alpha, K_\beta}$, which is denoted, by simplicity, as $\preceq_{\alpha,\beta}$ (see Huidobro et al. 2020).

Some of the orders we have considered as the lexicographical orders type 1 and 2 and the Xu and Yager order are particular cases of these admissible orders. Thus, $\preceq_{Lex1} \equiv \preceq_{0,1}$, $\preceq_{Lex2} \equiv \preceq_{1,0}$ and $\preceq_{YX} \equiv \preceq_{1/2,\beta}$ for any $\beta \in (1/2, 1]$ (see Bustince et al. 2013).

2.2 Inclusion

In the fuzzy set theory, we say that A is contained in B if and only if the membership function of A is less than or equal to the membership function of B, where $A, B \in FS(X)$ (see Zadeh 1965). Regarding IVFS, we propose the following definition of containment, which extends the fuzzy set definition.

Definition 3 (Huidobro et al. 2020) Let $(L([0, 1]), \preceq_o)$ be the set of all closed interval included in [0, 1] with any of the relations considered in the previous section. Let A and B be any sets in $IVFS(X)$, we say that A is o-included in B, which is denoted by $A \subseteq_o B$ if, and only if,

$$A(x) \preceq_o B(x), \forall x \in X.$$

It is clear that if $\preceq_o$ is an order in $L([0, 1])$, $\subseteq_o$ is an order in $IVFS(X)$. However, even in the case $\preceq_o$ is a total order, $\subseteq_o$ is just a partial order.

Example 1 Consider the IVFS A, B and C defined as in Figure 4, it is clear that $A, B \subseteq_{ID} C$ and therefore they are ID-included in C with respect to any of the considered orders. We also have $A \subseteq_{Lo} B$, but $A \not\subseteq_{ID} B$. Thus, A is included in B for any considered order except for the interval dominance. Finally, we can say that B or C are not included in A for any order.

As we commented, the inherited relation in $IVFS(X)$ is not a total order even in case $\preceq_o$ is a total order. Thus, if we consider $\subseteq_{Lex1}$ and the IVFS in Fig. 5, we have that A and B are not comparable by means of the order $\subseteq_{Lex1}$.

2.3 Embedding

At this point, it is important to remark that in the previous section we introduced the inclusion between IVFS based on a comparison between the membership values, as

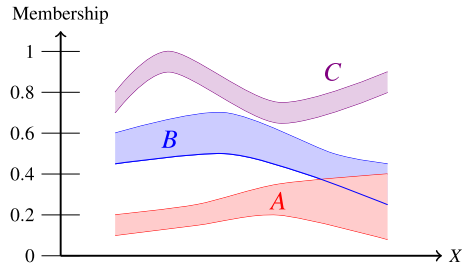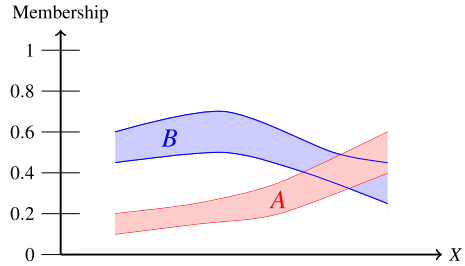**Fig. 4** Membership functions of A, B and C

Fig. 5 A and B are $\subseteq_{Lex1}$-incomparable

it holds for fuzzy sets. This comparison is made by studying if the membership value of a set at any point is "lower than or equal to" the membership value of the other set. We should notice difference to the comparison of membership values by means of the inclusion of intervals. In that case we are not measuring if the value is lower and so the set is included, but the precision on the definition of the IVFS.

Definition 4 Let A and B be any sets in $IVFS(X)$. Then A is embedded in B, which is denoted by $A \sqsubseteq_o B$ if, and only if,

$$A(x) \subseteq B(x), \forall x \in X$$

where $\subseteq$ is the usual inclusion between intervals.

Example 2 It is immediate from the definition that A is embedded in B if, and only if,

$$\underline{B(x)} \leq \underline{A(x)} \leq \overline{A(x)} \leq \overline{B(x)}, \forall x \in X.$$

If we consider the IVFS in Fig. 6a, we have that A is embedded in B, since $A(x) \subseteq B(x), \forall x \in X$.

The uncertainty about the real membership value for A is clearly lower than for B. We can also notice that, for example, A is not Lo-included in B.

We also have examples in the opposite direction. Thus, if we consider the IVFS in Fig. 6b, A is not embedded in B and B is not embedded in A, but $B \subset_{Lo} A$.

Thus, embedding is also a partial order in $IVFS(X)$, but its meaning is totally different from the idea behind the concept of inclusion of IVFS.
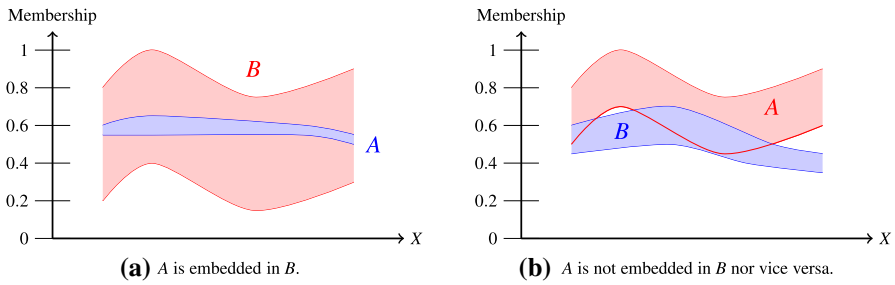
Membership

(a) *A is embedded in B.*

(b) *A is not embedded in B nor vice versa.*

Fig. 6 Is *A* "embedded" in *B*?

2.4 Complement

Several operations have been considered to this concept in the literature. We will consider now the most usual ones. Let us start with the simplest one.

Definition 5 (Dubois and Prade 2005) Let A be in $IVFS(X)$. The complement of A, denoted by A^c, is defined by $\underline{A^c}(x) = 1 - \overline{A}(x)$ and $\overline{A^c}(x) = 1 - \underline{A}(x)$ for any $x \in X$, that is,

$$A^c(x) = [1 - \overline{A}(x), 1 - \underline{A}(x)].$$

We can generalize this concept by means of a negation.

We are afraid that this should be replaced by:

Definition 6 (Gehrke et al. 2001) A function $N : [0, 1] \to [0, 1]$ is a negation if it is a one to one map such that N is decreasing and $N(N(x)) = x$.

By default we will consider the usual negation $N(x) = 1 - x$, for any $x \in X$.

3 Operations for IVFS

In this section we study the intersection and union of IVFS and define the concept of level sets for IVFS. The intersection and the level sets of IVFS be important notions for the study of convexity and the union is necessary to study the concept of a level set.

3.1 Intersection

In the literature, the intersection of two sets is defined as the greatest set that is contained in both sets, so we are going to apply this definition to IVFS. As we have seen, the chosen order matters, so we have a different definition of intersection for each one of the considered orders.

Definition 7 (Huidobro et al. 2020) Let A, B be IVFS in X and let $\preceq_o$ be an order in $L([0, 1])$. We define the o-intersection of A and B, denoted by $A \cap_o B$, as the greatest IVFS such that $A \cap_o B \subseteq_o A$ and $A \cap_o B \subseteq_o B$.

For any interval orders $\preceq_{o_1}$ and $\preceq_{o_2}$ in $IVFS(X)$ such that $a \preceq_{o_1} b$ implies that $a \preceq_{o_2} b$, for all $a, b \in L([0,1])$, we have that $A \cap_{o_1} B \subseteq_{o_2} A \cap_{o_2} B$ for any $A, B \in IVFS(X)$. We have to keep in mind the order relations in Fig. 3.

Taking into account the connection between the chosen orders, we will examine the obtained definition for each order, trying to join them in those cases where we can find a general behaviour: for the first group (interval dominance and lattice order) we consider partial order which defines the intersection as a unique set; for the second group (admissible orders and, in particular, the lexicographical orders and the Xu and Yager order) the intersection will be again defined uniquely but they are total orders; finally, for the third group (maximim, maximax, Hurwicz and weak orders) the intersection is to defined as a unique IVFS.

Let us start with the expression of the intersection by using interval dominance (ID) or lattice order (Lo).

Proposition 2 *Let $A, B \in IVFS(X)$. Then, for any $x \in X$ we have that*

- $A \cap_{ID} B(x) = \min\{\underline{A(x)}, \underline{B(x)}\}$.
- $A \cap_{Lo} B(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\overline{A(x)}, \overline{B(x)}\}]$.

Proof We start with the case of the interval dominance:

For any value x in X, it is clear $\min\{\underline{A(x)}, \underline{B(x)}\}$ is a number in [0, 1] and therefore an element in $L([0,1])$. Thus, if we consider the fuzzy set I defined as $I(x) = \min\{\underline{A(x)}, \underline{B(x)}\}$ for any $x \in X$, or equivalently the interval-valued fuzzy set defined as $I(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\underline{A(x)}, \underline{B(x)}\}]$ for any $x \in X$, we have that $\overline{I(x)} = \min\{\underline{A(x)}, \underline{B(x)}\} \leq \underline{A(x)}$ and $\overline{I(x)} \leq \underline{B(x)}$. Thus, $I(x) \preceq_{ID} A(x)$ and $I(x) \preceq_{ID} B(x)$ for any $x \in X$ and therefore $I \subseteq_{ID} A$ and $I \subseteq_{ID} B$.

Apart from that, if we consider a set $C \in IVFS(X)$ such that $C \subseteq_{ID} A$ and $C \subseteq_{ID} B$, then $\overline{C(x)} \leq \underline{A(x)}$ and $\overline{C(x)} \leq \underline{B(x)}$. So, $\overline{C(x)} \leq \min\{\underline{A(x)}, \underline{B(x)}\} = \overline{I(x)}$, that is, $C \subseteq_{ID} I$.

Thus, the fuzzy set I is the greatest interval-valued set that is ID-included in both sets and therefore it is the intersection of them.

Now, for the lattice order:

It is immediate that $[\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\overline{A(x)}, \overline{B(x)}\}] \in L([0,1])$ for any $x \in X$. Thus, we can defined an interval-valued fuzzy set I as follows: $I(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\overline{A(x)}, \overline{B(x)}\}]$ for all $x \in X$. Then, we have that $\underline{I(x)} = \min\{\underline{A(x)}, \underline{B(x)}\} \leq \underline{A(x)}$ and $\overline{I(x)} = \min\{\overline{A(x)}, \overline{B(x)}\} \leq \overline{A(x)}$. Thus, $I(x) \preceq_{Lo} A(x)$ and therefore $I \subseteq_{Lo} A$. Similarly, we can prove that $I \subseteq_{Lo} B$.

Finally, if we consider a set $C \in IVFS(X)$ such that $C \subseteq_{Lo} A$ and $C \subseteq_{Lo} B$, then $\underline{C(x)} \leq \underline{A(x)}$ and $\underline{C(x)} \leq \underline{B(x)}$. Therefore, $\underline{C(x)} \leq \min\{\underline{A(x)}, \underline{B(x)}\} = \underline{I(x)}$. It is analogous to prove that $\overline{C(x)} \leq \min\{\overline{A(x)}, \overline{B(x)}\} = \overline{I(x)}$ and then $C \subseteq_{Lo} I$.

Thus, the interval-valued fuzzy set I is the greatest, w.r.t. the lattice order, interval-valued such that it is Lo-included in A and B and therefore, by definition, I is the Lo-intersection of them. □

Notice that the intersection of two IVFS by means of the interval dominance is just a fuzzy set and that the expression obtained for the lattice order is the most usual intersection considered in the literature.

For the case of the lexicographical orders and the Xu and Yager order, they are particular cases of admissible order and the intersection is obtained as a consequence of the following general result for total orders.

Proposition 3 *Let $\preceq_o$ be a total order on $L([0, 1])$. For any $A, B \in IVFS(X)$, the o-intersection of A and B is the IVFS defined by:*

$$A \cap_o B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_o B(x), \\ B(x) & \text{if } B(x) \preceq_o A(x). \end{cases}$$

Proof It is clear that the set defined in the statement is an interval-valued fuzzy set. Let us denote it by I. We have that $I(x) = A(x)$ if $A(x) \preceq_o B(x)$ and $I(x) = B(x)$ if $B(x) \preceq_o A(x)$. Since $\preceq_o$ is transitive, we have that $I(x) \preceq_o A(x)$ and $I(x) \preceq_o B(x)$ for any $x \in X$. Thus, $I \subseteq_o A$ and $I \subseteq_o B$.

Moreover, if we consider a set $C \in IVFS(X)$ such that $C \subseteq_o A$ and $C \subseteq_o B$, then for any $x \in X$, as under $\preceq_o$ there are not incomparable elements, we have two cases:

(a) If $A(x) \preceq_o B(x)$, then $I(x) = A(x)$. But, $C(x) \preceq_o A(x) = I(x)$.
(b) If $B(x) \preceq_o A(x)$, then $I(x) = B(x)$. But, $C(x) \preceq_o B(x) = I(x)$.

Thus, in both case we obtain that $C(x) \preceq_o I(x)$ and therefore $C \subseteq_o I$.

Therefore I is the greatest interval-valued fuzzy set o-included in A and B and, by definition, it is its o-intersection. □

Thus, if $\preceq_{\mathscr{A},\mathscr{B}}$ is an admissible order as the ones considered in Proposition 1, we have that

$$A \cap_{\mathscr{A},\mathscr{B}} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{\mathscr{A},\mathscr{B}} B(x), \\ B(x) & \text{if } B(x) \preceq_{\mathscr{A},\mathscr{B}} A(x), \end{cases}$$

and, in particular:

– Lexicographical order type 1:

$$A \cap_{Lex1} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{Lex1} B(x), \\ B(x) & \text{if } B(x) \preceq_{Lex1} A(x). \end{cases}$$

– Lexicographical order type 2:

$$A \cap_{Lex2} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{Lex2} B(x), \\ B(x) & \text{if } B(x) \preceq_{Lex2} A(x). \end{cases}$$

– Xu and Yager order:

$$A \cap_{YX} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{YX} B(x), \\ B(x) & \text{if } B(x) \preceq_{YX} A(x). \end{cases}$$

We have just seen that when using interval dominance, lattice order or any of the admissible orders, we obtain a unique intersection. Unluckily, not all the orders considered have the same behavior as we can see in the following result.

Table 2 Properties of the intersection

Interval order	Is the intersection unique?	Is the intersection an IVFS?
Interval dominance	✓	✗
Lattice order	✓	✓
Lex. order type 1	✓	✓
Lex. order type 2	✓	✓
Xu and Yager order	✓	✓
Maximim order	✗	
Maximax order	✗	
Hurwicz order	✗	
Weak order	✗	

Proposition 4 *Let $A, B \in IVFS(X)$. Then, for any $x \in X$,*

- *Maximim order: $A \cap_{Mm} B(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, v]$ where v can be any number in the interval $[\min\{\underline{A(x)}, \underline{B(x)}\}, 1]$.*
- *Maximax order: $A \cap_{MM} B(x) = [u, \min\{\overline{A(x)}, \overline{B(x)}\}]$ where u can be any number in the interval $[0, \min\{\overline{A(x)}, \overline{B(x)}\}]$.*
- *Hurwicz order: $A \cap_{H(\alpha)} B(x) = \left[u, \frac{k - \alpha \cdot u}{1 - \alpha}\right]$ where $k = \min\{\alpha \cdot \underline{A(x)} + (1 - \alpha) \cdot \overline{A(x)}, \alpha \cdot \underline{B(x)} + (1 - \alpha) \cdot \overline{B(x)}\}$ and u is any number in the interval $\left[\max\{0, \frac{k - (1 - \alpha)}{\alpha}\}, k\right]$.*
- *Weak order: $A \cap_{wo} B(x) = [u, v]$ where u could be any number in the interval $[0, \min\{\underline{A(x)}, \underline{B(x)}\}]$ and v can be any number in the interval $[\min\{\overline{A(x)}, \overline{B(x)}\}, 1]$.*

Proof At the first two cases (Mm and MM), it is immediate to check that this set is included in A and B and it is the greatest element of $IVFS(X)$ fulfilling this property.

For the Hurwicz order, the intersection is well-defined, since $u \geq 0$, $u \leq \frac{k - \alpha \cdot u}{1 - \alpha}$ iff $u \leq k$ and this is true by definition and, finally, $\frac{k - \alpha \cdot u}{1 - \alpha} \leq 1$ iff $u \geq \frac{k - (1 - \alpha)}{\alpha}$ and this is also true by definition. Moreover, since $\alpha u + (1 - \alpha)\frac{k - \alpha \cdot u}{1 - \alpha} = k$, we have that the defined set is $H(\alpha)$-included in A and B. Moreover, if we have any other set C such that $C \subseteq_{H(\alpha)} A$ and $C \subseteq_{H(\alpha)} B$, then, for any $x \in X$, $\alpha \underline{C(x)} + (1 - \alpha)\overline{C(x)} \leq k = \alpha u + (1 - \alpha)\frac{k - \alpha \cdot u}{1 - \alpha}$. Thus, there is not an interval-valued fuzzy sets $H(\alpha)$-included in A and B which is not included in the set defined at the statement.

For the weak order, since $u \leq \min\{\underline{A(x)}, \underline{B(x)}\}$, then $[u, v] \preceq_{wo} A(x)$ and $[u, v] \preceq_{wo} B(x)$ and any other set C such that $C \subseteq_{wo} A$ and $C \subseteq_{wo} B$ fulfils that $\underline{C(x)} \leq \underline{A(x)}$ and $\underline{C(x)} \leq \underline{B(x)}$, that is, $\underline{C(x)} \leq \min\{\underline{A(x)}, \underline{B(x)}\} \leq v$. □

Taking into account the previous result, we can see that in some cases the intersection is not uniquely defined. For the interval dominance, we have that the intersection of two IVFS is just a fuzzy set (see Table 2).

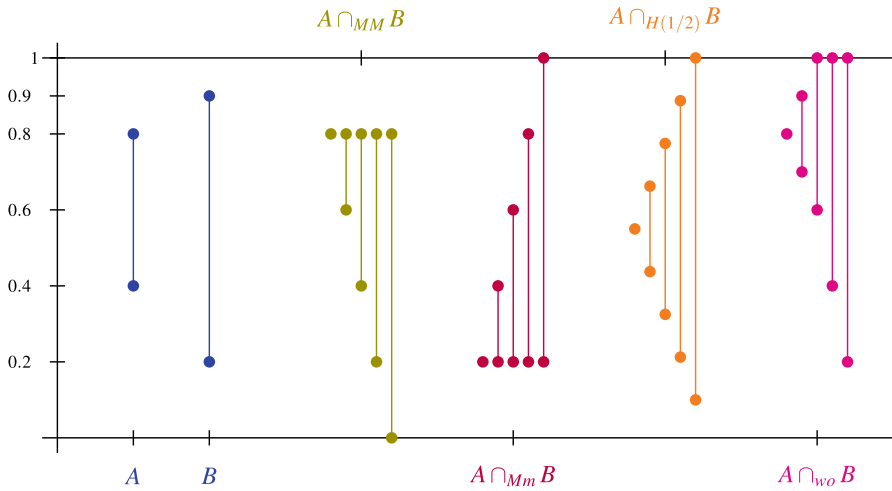We can clarify the previous comments by means of the following examples.

Fig. 7 Not unique intersections of A and B

Example 3 Let us consider the case $X = \{x\}$ and the IVFS A and B defined by $A(x) = [0.4, 0.8]$ and $B(x) = [0.2, 0.9]$. Then, the intersection for some of the orders is:

$A \cap_{MM} B(x)$	$A \cap_{Mm} B(x)$	$A \cap_{H(1/2)} B(x)$	$A \cap_{wo} B(x)$
$[u, 0.8]$	$[0.2, v]$	$[u, 1.1 - u]$	$[u, v]$
$u \in [0, 0.8]$	$v \in [0.2, 1]$	$u \in [0.1, 0.55]$	$u \in [0, 0.8]$
			$v \in [0.8, 1]$

which is graphically represented in Fig. 7.

If we consider the orders where the intersection is unique, we obtain that:

$A \cap_{ID} B$	$A \cap_{Lo} B$	$A \cap_{Lex1} B$	$A \cap_{Lex2} B$	$A \cap_{YX} B$
0.2	$[0.2, 0.8]$	$[0.2, 0.9]$	$[0.4, 0.8]$	$[0.2, 0.9]$

where we can see that the intersection is just a fuzzy set for the case of the interval dominance. These examples are graphically represented in Fig. 8.

In this case the lexicographical order type 1 and the Xu and Yager order give the same intersection but, of course, this is not true in general. For example, if we consider C an IVFS such that $C(x) = [0.4, 0.5]$, we have that $B \preceq_{Lex1} C$ and $C \preceq_{YX} B$ and therefore $B \cap_{Lex1} C = B \neq B \cap_{YX} C = C$.

It is also clear, from this example that the intersection depends on the considered order. From now on, we will only consider the orders where the intersection is uniquely defined, that is, where is it a unique set.
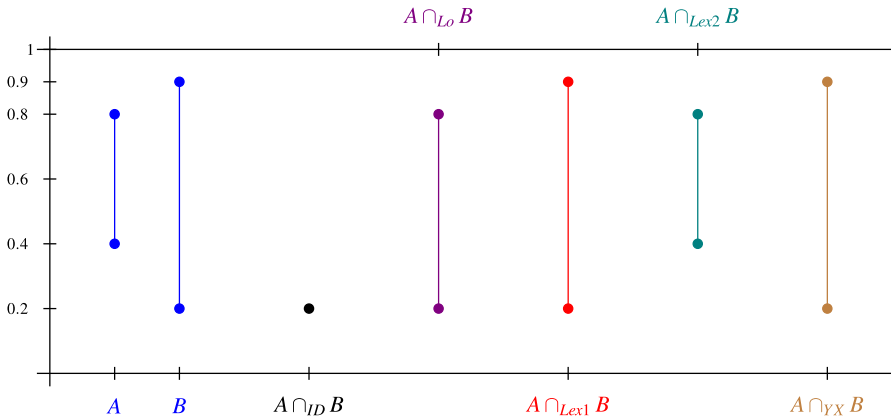
Fig. 8 Intersection of A and B by different orders

3.2 Union

If the union of two sets is defined as the smallest set that contains both sets, then we have a different definition of union for each order we are considering in $IVFS(X)$. So we can perform an analogous study to the intersection, as the union would be an important tool for the next section.

Definition 8 Let $A, B \in IVFS(X)$ and let $\preceq_o$ be an order in $L([0, 1])$. We define the o-union of A and B, denoted by $A \cup_o B$, as the smallest IVFS such that $A \subseteq_o A \cup_o B$ and $B \subseteq_o A \cup_o B$.

Thus, we are going to consider only the orders where the intersection is unique and we will follow an analogous scheme for the intersection.

Proposition 5 *Let $A, B \in IVFS(X)$. Then, for any $x \in X$ we have:*

- $A \cup_{ID} B(x) = \max\{\overline{A(x)}, \overline{B(x)}\}.$
- $A \cup_{Lo} B(x) = [\max\{\underline{A(x)}, \underline{B(x)}\}, \max\{\overline{A(x)}, \overline{B(x)}\}].$

Proof Interval dominance (ID):

We should prove that $A \subseteq A \cup_{ID} B$ and $B \subseteq A \cup_{ID} B$ and that if there is another IVFS containing both of them, then the union is contained in it. It is immediate that $A \subseteq A \cup_{ID} B$ and $B \subseteq A \cup_{ID} B$ by definition. Let us suppose there is an IVFS C such that $A \subseteq C$ and $B \subseteq C$. If $A \subseteq C$, then $\overline{A(x)} \leq \overline{C(x)}$. If $B \subseteq C$, thus $\overline{B(x)} \leq \overline{C(x)}$. If $A \cup_{ID} B(x) = \max\{\overline{A(x)}, \overline{B(x)}\}$, then $\max\{\overline{A(x)}, \overline{B(x)}\} \leq \underline{C(x)}$ and $A \cup_{ID} B \subseteq C$.

Lattice order (Lo):

Let us check that this union is well defined. First of all, we prove that $A \subseteq A \cup_{Lo} B$ and $B \subseteq A \cup_{Lo} B$. It is fulfilled by definition. If we suppose that there exists an interval valued fuzzy set $C \in IVFS(X)$, $C(x) = [\underline{C(x)}, \overline{C(x)}]$, such that $A \subseteq C$, $B \subseteq C$ and $C \subset A \cup_{Lo} B$. If $A \subseteq C$, then $\underline{A(x)} \leq \underline{C(x)}$ and $\overline{A(x)} \leq \overline{C(x)}$. If $B \subseteq C$, thus $\underline{B(x)} \leq \underline{C(x)}$ and $\overline{B(x)} \leq \overline{C(x)}$. Then $A \cup_{Lo} B \subseteq C$, so $A \cup_{Lo} B = C$. □

Similarly to the case of the intersection, for the union, the interval dominance gives us again a fuzzy set as it is just one point. The usual union considered in the literature is again the one given by the lattice order.

For the case of total orders, we have that:

Proposition 6 *Let $\preceq_o$ be a total order on $L([0, 1])$. For any $A, B \in IVFS(X)$, the o-union of A and B is the IVFS defined by:*

$$A \cup_o B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_o B(x), \\ A(x) & \text{if } B(x) \preceq_o A(x). \end{cases}$$

Proof Let us check that this union is well defined. Since $\preceq_o$ is a total order, it is immediate that $A \cup_o B$ is defined for any $x \in X$, since $A(x) \preceq_o B(x)$ or $B(x) \preceq_o A(x)$.

By definition, it is clear that $A \subseteq A \cup_o B$ and $B \subseteq A \cup_o B$.

Finally, if we suppose that there exists an interval valued fuzzy set $C \in IVFS(X)$ such that $A \subseteq_o C$ and $B \subseteq_o C$, then, by the transitivity of $\preceq_o$, $A(x) \preceq_o C(x)$ and $B(x) \preceq_o C(x)$, for any $x \in X$. Then, by definition, it is immediate that $A \cup_o B(x) \preceq_o C(x)$ and therefore $A \cup_o B \subseteq_o C$. □

Thus, for the admissible order considered in Proposition 1 we have that

$$A \cup_{\mathscr{A},\mathscr{B}} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{\mathscr{A},\mathscr{B}} B(x), \\ A(x) & \text{if } B(x) \preceq_{\mathscr{A},\mathscr{B}} A(x), \end{cases}$$

and, in particular:

– Lexicographical order type 1:

$$A \cup_{Lex1} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{Lex1} B(x), \\ A(x) & \text{if } B(x) \preceq_{Lex1} A(x). \end{cases}$$

– Lexicographical order type 2:

$$A \cup_{Lex2} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{Lex2} B(x), \\ A(x) & \text{if } B(x) \preceq_{Lex2} A(x). \end{cases}$$

– Xu and Yager order:

$$A \cup_{YX} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{YX} B(x), \\ A(x) & \text{if } B(x) \preceq_{YX} A(x). \end{cases}$$

We present an example for a better understanding of this operation.

Example 4 Under the same conditions of Example 3, the union is:

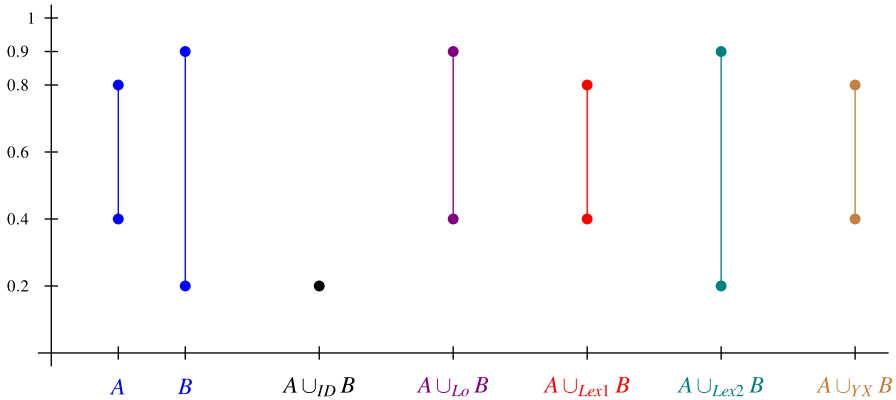| $A \cup_{ID} B$	$A \cup_{Lo} B$	$A \cup_{Lex1} B$	$A \cup_{Lex2} B$	$A \cup_{YX} B$
0.9	[0.4, 0.9]	[0.4, 0.8]	[0.2, 0.9]	[0.4, 0.8]

Fig. 9 Union of A and B by different orders

where we can see that the intersection is just a fuzzy set for the case of the interval dominance. Graphically this can be seen in Fig. 9.

In this case, for the lattice order, A and B are incomparable, however, $B \subseteq_{Lex1} A$, $A \subseteq_{Lex2} B$ and $B \subseteq_{YX} A$. Thus, it is logical that $A \cup_{Lo} B \neq A$ and $A \cup_{Lo} B \neq B$, $A \cup_{Lex1} B = A$, $A \cup_{Lex2} B = B$ and $A \cup_{YX} B = A$. We can notice again the strong influence of the order considered to define the operation of inclusion in $IVFS(X)$, as it is natural, seems the union is very related to this concept.

As in the intersection, if we use the lexicographical order type 1 or the Xu and Yager order we obtain the same IVFS but, in general, this is not true. For instance, if we consider C an IVFS such that $C(x) = [0.5, 0.6]$, we have that $C \preceq_{Lex1} A$ and $A \preceq_{YX} C$ and therefore $A \cup_{Lex1} C = A \neq A \cup_{YX} C = C$.

3.3 Level sets of interval-valued fuzzy sets

One of the most important concepts of fuzzy sets is an α-cut or a level set (Klir and Yuan 1995). In this section we give a suitable definition of a level set for IVFS.

Definition 9 Let $\preceq_o$ be an order (reflexive, symmetric and transitive) on $L([0, 1])$. For any $A \in IVFS(X)$ and for any $[\alpha, \beta] \in L([0, 1])$, we define the $[\alpha, \beta]$-level sets of A w.r.t. the order $\preceq_o$ as it follows:

$$A_{[\alpha,\beta]}^o = \{x \in X : [\alpha, \beta] \preceq_o A(x)\}.$$

As the definition depends on the order we are using, it is interesting that we would obtain different level sets if we use different orders.

Example 5 Let $X = \{x, y, z\}$. If we consider $A \in IVFS(X)$ defined as follows: we have calculated some level sets for different orders in Table 3:

X	x	y	z
A	[0.1, 0.7]	[0.2, 0.8]	[0.4, 0.5]

Table 3 Level sets for different orders

Order	$A_{[0.1,0.7]}$	$A_{[0.2,0.8]}$	$A_{[0.4,0.5]}$
Interval dominance	$\emptyset$	$\emptyset$	$\emptyset$
Lattice order	$\{x, y\}$	$\{y\}$	$\{z\}$
Lexicographical type 1	$\{x, y, z\}$	$\{y, z\}$	$\{z\}$
Lexicographical type 2	$\{x, z\}$	$\{y\}$	$\{x, y, z\}$
Xu and Yager	$\{x, y, z\}$	$\{y\}$	$\{y, z\}$

An equivalent definition for level sets, for the particular case of the lattice order, was considered in Ramík and Vlach (2016) for intuitionistic fuzzy sets. The equivalence is a consequence of the mathematical relationship between interval-valued fuzzy sets and intuitionistic fuzzy sets.

Based on this example, we can notice that some level sets are included in others because of the relation between those orders.

Proposition 7 *If $\preceq_1$ and $\preceq_2$ are orders in $L([0, 1])$ such that $a \preceq_1 b$ implies $a \preceq_2 b$, then for any $A \in IVFS(X)$ and any $[\alpha, \beta] \in L([0, 1])$ we have that $A^1_{[\alpha,\beta]} \subseteq A^2_{[\alpha,\beta]}$.*

Proof By definition, $A^1_{[\alpha,\beta]} = \{x \in X : [\alpha, \beta] \preceq_1 A(x)\} \subseteq \{x \in X : [\alpha, \beta] \preceq_2 A(x)\} = A^2_{[\alpha,\beta]}$. □

For instance, in Example 5, the level sets of A using interval dominance are included in the level sets using the lattice order, which are also included in the ones obtained using the lexicographical order type 1, type 2 or the Xu and Yager order.

Let us overview some of the properties that these level sets fulfill.

Proposition 8 *Let $\preceq_o$ be an order (reflexive, symmetric and transitive) on $L(]0, 1])$. For any $A, B \in IVFS(X)$ and any $[\alpha, \beta], [\gamma, \delta] \in L([0, 1])$, we have that:*

(i) *If $[\alpha, \beta] \preceq_o [\gamma, \delta]$, then $A^o_{[\gamma,\delta]} \subseteq A^o_{[\alpha,\beta]}$.*
(ii) *$A \subseteq_o B \Leftrightarrow A^o_{[\alpha,\beta]} \subseteq B^o_{[\alpha,\beta]}$ for any $[\alpha, \beta] \in L(]0, 1])$.*
(iii) *$(A \cap_o B)^o_{[\alpha,\beta]} \subseteq A^o_{[\alpha,\beta]} \cap B^o_{[\alpha,\beta]}$. If $\preceq_o$ is a total order, then $(A \cap_o B)^o_{[\alpha,\beta]} = A^o_{[\alpha,\beta]} \cap B^o_{[\alpha,\beta]}$.*
(iv) *$A^o_{[\alpha,\beta]} \cup B^o_{[\alpha,\beta]} \subseteq_o (A \cup_o B)^o_{[\alpha,\beta]}$. If $\preceq_o$ is a total order, then $A^o_{[\alpha,\beta]} \cup B^o_{[\alpha,\beta]} = (A \cup_o B)^o_{[\alpha,\beta]}$.*

Proof Let us consider $A, B \in IVFS(X)$ and $[\alpha, \beta], [\gamma, \delta] \in L([0, 1])$.

(i) If $[\alpha, \beta] \preceq_o [\gamma, \delta]$, then it is immediate by definition that $A^o_{[\gamma,\delta]} \subseteq A^o_{[\alpha,\beta]}$, since $\preceq_o$ is transitive.
(ii) If $A \subseteq_o B$ then $A(x) \preceq_o B(x), \forall x \in X$. Thus, if $[\alpha, \beta] \preceq_o A(x)$, since $\preceq_o$ is transitive, then $[\alpha, \beta] \preceq_o B(x)$ and so $A^o_{[\alpha,\beta]} = \{x \in X : [\alpha, \beta] \preceq_o A(x)\} \subseteq \{x \in X : [\alpha, \beta] \preceq_o B(x)\} = B^o_{[\alpha,\beta]}$.

Conversely, for any $x \in X$, if we apply the inclusion for the $A(x)$-level sets, we have that $x \in A^0_{A(x)}$ since $\preceq_o$ is reflexive, and therefore $x \in B^0_{A(x)}$. This is equivalent to say that $A(x) \preceq_o B(x)$. As we have proven it for any $x \in X$ we have that $A \subseteq_o B$.

(iii) Since $A \cap_o B \subseteq_o A$ and $A \cap_o B \subseteq_o B$, by applying ii), we have that $(A \cap_o B)^o_{[\alpha,\beta]} \subseteq A^o_{[\alpha,\beta]}$ and $(A \cap_o B)^o_{[\alpha,\beta]} \subseteq B^o_{[\alpha,\beta]}$ and therefore $(A \cap_o B)^o_{[\alpha,\beta]} \subseteq A^o_{[\alpha,\beta]} \cap B^o_{[\alpha,\beta]}$. Conversely, if $x \in A^o_{[\alpha,\beta]} \cap B^o_{[\alpha,\beta]}$, then $[\alpha,\beta] \preceq_o A(x)$ and $[\alpha,\beta] \preceq_o B(x)$. As we are using a total order, from Proposition 6 we have that $A \cap B(x) = A(x)$ or $A \cap B(x) = B(x)$ and so $[\alpha,\beta] \preceq_o A \cap B(x)$.

(iv) Since $A \subseteq_o A \cup_o B$ and $B \subseteq_o A \cup_o B$, by applying ii), we have that $A^o_{[\alpha,\beta]} \subseteq_o (A \cup_o B)^o_{[\alpha,\beta]}$ and $B^o_{[\alpha,\beta]} \subseteq_o (A \cup_o B)^o_{[\alpha,\beta]}$. Then, $A^o_{[\alpha,\beta]} \cup B^o_{[\alpha,\beta]} \subseteq_o (A \cup_o B)^o_{[\alpha,\beta]}$. Conversely, for any $x \in X$ we have that $A \cup_o B(x) = B(x)$ or $A \cup_o B(x) = B(x)$, by applying Proposition 6, since $\preceq_o$ is a total order. Thus, if $x \in (A \cup_o B)^o_{[\alpha,\beta]}$, then $[\alpha,\beta] \preceq_o A \cup B(x)$ and therefore $[\alpha,\beta] \preceq_o A(x)$ or $[\alpha,\beta] \preceq_o B(x)$. Then, $x \in A^o_{[\alpha,\beta]} \cup B^o_{[\alpha,\beta]}$.

$\square$

In fuzzy sets theory, we can represent a fuzzy set by its α-cuts through the Decompositions Theorems from Klir and Yuan (1995), so the next task we consider is adapting these results of fuzzy sets into IVFS. Thus, we will try to identify an IVFS through its level sets. First of all, we will do it in an example and then we will prove a general result.

Example 6 Let $X = \{x, y, z\}$. If we consider the IVFS A defined in Example 5 and the lexicographical order type 1, then the level sets are

$$A^{Lex1}_{[0.1,0.7]} = \{x, y, z\},$$
$$A^{Lex1}_{[0.2,0.8]} = \{y, z\}$$

and

$$A^{Lex1}_{[0.4,0.5]} = \{z\}.$$

If we choose proper intervals, the IVFS can be represented by its level sets. Let us use the following characteristic functions to define the level sets:

$$A^{Lex1}_{[0.1,0.7]} = 1 \cdot \{x\} + 1 \cdot \{y\} + 1 \cdot \{z\} = \{x, y, z\},$$
$$A^{Lex1}_{[0.2,0.8]} = 0 \cdot \{x\} + 1 \cdot \{y\} + 1 \cdot \{z\} = \{y, z\}$$

and

$$A^{Lex1}_{[0.4,0.5]} = 0 \cdot \{x\} + 0 \cdot \{y\} + 1 \cdot \{z\} = \{z\}.$$

Now, we are going to obtain IVFS based on these level sets defined as follows:

$$_{[\alpha,\beta]}^{Lex1}A = [\alpha, \beta] \cdot A^{Lex1}_{[\alpha,\beta]} = \begin{cases} [\alpha, \beta] & \text{if } x \in A^{Lex1}_{[\alpha,\beta]}, \\ [0, 0] & \text{otherwise.} \end{cases}$$

With this operation, we are interval-valued fuzzifying the level sets, that is, we start from level sets (crisp sets) and we get IVFS.

Then,

$$\underset{[0.1,0.7]}{Lex1}A(t) = [0.1, 0.7], \forall t \in X,$$

$$\underset{[0.2,0.8]}{Lex1}A(t) = \begin{cases} [0.2, 0.8] & \text{if } t \in \{y, z\}, \\ [0, 0] & \text{if } t = x, \end{cases}$$

and

$$\underset{[0.4,0.5]}{Lex1}A(t) = \begin{cases} [0.4, 0.5] & \text{if } t = z, \\ [0, 0] & \text{if } t \in \{x, y\}. \end{cases}$$

That is how the previous notation works, when we have one interval fixed, if we obtain that interval in any of the level set functions, it means that the element belongs to that level set, as we can see here:

$\underset{[0.1,0.7]}{Lex1}A(x) = [0.1, 0.7], \underset{[0.1,0.7]}{Lex1}A(y) = [0.1, 0.7]$ and $\underset{[0.1,0.7]}{Lex1}A(z) = [0.1, 0.7],$

$\underset{[0.2,0.8]}{Lex1}A(y) = [0.2, 0.8]$ and $\underset{[0.2,0.8]}{Lex1}A(z) = [0.2, 0.8],$

$\underset{[0.4,0.5]}{Lex1}A(z) = [0.4, 0.5].$

It is immediate that the $Lex1 - union$ of these IVFS is the original set A. That is,

$$A = \underset{[0.1,0.7]}{Lex1}A \cup_{Lex1} \underset{[0.2,0.8]}{Lex1}A \cup_{Lex1} \underset{[0.4,0.5]}{Lex1}A.$$

Based on this idea, we propose the following theorem:

Theorem 1 (Decomposition Theorem) *Let $\preceq_o$ be a total order in $L([0, 1])$. For every $A \in IVFS(X)$, we have that*

$$A = \underset{[\alpha,\beta] \in L([0,1])}{\cup_o} {}_{[\alpha,\beta]}^{o}A,$$

where $\cup_o$ denotes the $o - union$ and ${}_{[\alpha,\beta]}^{o}A(x) = [\alpha, \beta]$ if $x \in A_{[\alpha,\beta]}^{o}$ and 0 otherwise.

Proof Let A be any set in $IVFS(X)$. For any $x \in X$, we have that $A(x) = [\gamma, \delta] \in L([0, 1])$. Thus, $A(x) = {}_{[\gamma,\delta]}^{o}A(x)$ and therefore $A(x) \preceq_o \underset{[\alpha,\beta] \in L([0,1])}{\cup_o} {}_{[\alpha,\beta]}^{o}A(x)$, by the definition of $\cup_o$.

On the other hand, since $\preceq_o$ is a total order, we have that $\cup_o {}_{[\alpha,\beta] \in L([0,1])} {}_{[\alpha,\beta]}A(x) = {}_{[\epsilon,\zeta]}^{o}A(x)$ for some $[\epsilon, \zeta] \in L([0, 1])$.

By the definition of ${}_{[\epsilon,\zeta]}^{o}A(x)$, we have two cases:

- If $x \notin A_{[\epsilon,\zeta]}^{o}$, then ${}_{[\epsilon,\zeta]}^{o}A(x) = [0, 0] \preceq_o A(x)$.
- If $x \in A_{[\epsilon,\zeta]}^{o}$, then $[\epsilon, \delta] \preceq_o A(x)$ and so ${}_{[\epsilon,\zeta]}^{o}A(x) = [\epsilon, \zeta] \preceq_o A(x)$.

Thus, by the symmetry of $\preceq_o$, we have that $A(x) = \cup_o {}_{[\alpha,\beta] \in L([0,1])} {}_{[\alpha,\beta]}A(x)$. □

This theorem allows us to work with level sets instead of the IVFS, but not all the operations can be applied. For instance, the standard complement for IVFS is not cutworthy, that is,

$$(A^c)^o_{[\alpha,\beta]} \neq (A^o_{[\alpha,\beta]})^c,$$

as we can see in the following example.

Example 7 Under the same conditions of Example 5 we have that,

X	x	y	z
A	[0.1, 0.7]	[0.2, 0.8]	[0.4, 0.5]
A^c	[0.3, 0.9]	[0.2, 0.8]	[0.5, 0.6]

If we consider again the lexicographical order type 1 and the level [0.3, 0.9], we obtain that $(A^c)^{Lex1}_{[0.3,0.9]} = \{t \in X : [0.3, 0.9] \preceq_{Lex1} A^c(t)\} = \{x, z\}$.

On the other hand, $A^{Lex1}_{[0.3,0.9]} = \{z\}$ and then

$$\left(A^{Lex1}_{[0.3,0.9]}\right)^c = \{x, y\} \neq (A^c)^{Lex1}_{[0.3,0.9]}.$$

It is interesting that different intervals could generate the same level set, so we are going to take it into account in the next corollary. If we consider $\Lambda(A) = \{A(x) : x \in X\}$, there is a equivalent relation in X because $\Lambda(A)$ is the set of all intervals that represent different level sets of A. So the next result is a version of the first one where we only take one interval from each equivalent class in $\Lambda(A)$. That is, instead of considering $L([0, 1])$, we would use $\Lambda(A)$. In Example 5, $\Lambda(A) = \{[0.1, 0.7], [0.2, 0.8], [0.4, 0.5]\}$.

Corollary 1 *Let $\preceq_o$ be a total order in $L([0, 1])$. For every $A \in IVFS(X)$,*

$$A = \bigcup_{[\alpha,\beta]\in\Lambda(A))} {}_{[\alpha,\beta]}^o A.$$

This is the way to represent IVFS without repeating a level set. The proof is a consequence of the Decomposition Theorem.

4 Convexity of IVFS

Keeping in mind the comments of the previous section, we do not consider all the orders. As we could see, the intersection based on the maximax, the maximin, the Hurwicz or the weak order does not work well. For the remaining intersections, we will check if the intersection of two convex IVFS is a convex set as well.

Thus, first of all, we will remember the concept of convexity for interval-valued fuzzy sets. This idea arised from the starting point of the study of the convexity for fuzzy sets. It is well-known that Zadeh (1965) defined a fuzzy set μ_A in X by means of the map $\mu_A : X \to [0, 1]$. For any $\alpha \in (0, 1]$, there is a crisp subset of X associated to A in the following way: $(\mu_A)_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$. This set is known as the α-cut of μ_A and the collection of all the alpha-cuts completely characterizes the fuzzy set.

A particular case of fuzzy sets is convex fuzzy sets. A crisp subset A of a linear space X is convex if and only if $\lambda x + (1-\lambda)y \in A$ for any $x, y \in A$ and for any $\lambda \in [0, 1]$ (for a detailed study on convex set see e.g. Lay (2007)). Based on this definition, a natural extension for fuzzy sets could be: $\mu_A(\lambda x + (1-\lambda)y) \geq \lambda\mu_A(x) + (1-\lambda)\mu_A(y)$, for all $x, y \in X$ and for all $\lambda \in [0, 1]$. However, this definition was discarded as Zadeh (1965) pointed out that there is not an equivalence between convexity of fuzzy sets and convexity of the alpha-cuts.

Zadeh (1965) proposed the first definition of a convex fuzzy set as a fuzzy set that fulfills $\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for any $x, y \in X$ and any $\lambda \in [0, 1]$. He also proof that a fuzzy set is convex if and only if the α-cuts are convex crisp sets, for any $\alpha \in (0, 1]$, that is, the cutworthy property is verified. Apart from that, he was able to show that if A and B are two convex fuzzy sets, then $A \cap B$ is a convex fuzzy set. Another point to take into account this definition is that the addition appears on the right side of the inequality and it could make no sense when working in a more general environment (e.g. IVFS), where such operation is not defined in general.

In the literature, there are some approaches to convex interval-valued functions as Cao (2009). However, they are not dealing with IVFS, so we will consider the following definition of convexity that do not have the problem of defining the addition for IVFS. Thus, our starting point was the classical idea of convex set.

Definition 10 (Huidobro et al. 2020) Let X be an ordered space and let $\preceq_o$ be an order in $L([0, 1])$. An interval-valued fuzzy set A on X is said to be o-convex, if for each $x < y < z$ in X the following inequalities are fulfilled:

$$A(x) \preceq_o A(y) \text{ or } A(z) \preceq_o A(y).$$

This definition is based on the usual idea of convexity. It is clear that this definition depends on the order considered on $L([0, 1])$. Thus, this is a very flexible definition, which can be adapted to the necessities of the possible user, by considering the ranking methods of interval numbers more appropriate for her/his proposals. Of course, a order could be fixed for all the study, but this is the most general possible definition. In fact, it is easy to prove that if we consider a convex fuzzy set as an interval-valued fuzzy set, it is convex w.r.t. the previous definition for any order. In addition, this definition has as particular cases the the usual definition of convexity for crisp sets and fuzzy sets, independent on the considered order.

Remark 1 When X is a total ordered space, the previous definition of convexity is equivalent to check

$$\min\{A(x), A(z)\} \preceq_o A(y).$$

If we work with partial orders, it may happen that $A(x)$ is not related to $A(z)$, so this is the reason for considering $A(x) \preceq_o A(y)$ or $A(z) \preceq_o A(y)$ at in previous definition.

We would like to introduce the notion of strictly convex IVFS, based on this concept.

Definition 11 Let X be an ordered space and let $\preceq_o$ be an order on $L([0, 1])$. An interval-valued fuzzy set A on X is said to be strictly o-convex, if for each $x < y < z$ in X the following inequalities are fulfilled:

$$A(x) \prec_o A(y) \text{ or } A(z) \prec_o A(y)$$

which means that

$$A(x) \preceq_o A(y) \text{ and } A(x) \neq A(y)$$

or

$$A(z) \preceq_o A(y) \text{ and } A(z) \neq A(y).$$

Definition 10 is a proper definition because there is an equivalent between convexity and convexity of the level sets, as we can see in the following proposition.

Proposition 9 *Let X be a totally ordered space and let $A \in IVFS(X)$ and let $\preceq_o$ be an order in $L([0, 1])$. If A is a o-convex IVFS, then $A^o_{[\alpha,\beta]}$ are convex crisp sets for all $[\alpha, \beta] \in L([0, 1])$. The converse is true if $\preceq_0$ is a total order.*

Proof Let us consider $x, y, z \in X$ such that $x < y < z$.

If $x \in A^o_{[\alpha,\beta]}$ and $z \in A^o_{[\alpha,\beta]}$, then $[\alpha, \beta] \preceq_o A(x)$ and $[\alpha, \beta] \preceq_o A(z)$. Moreover, as A is convex, we have $A(x) \preceq_o A(y)$ or $A(z) \preceq_o A(y)$. By the transitivity of $\preceq_o$, $[\alpha, \beta] \preceq_o A(y)$ or $[\alpha, \beta] \preceq_o A(y)$ and so $y \in A^o_{[\alpha,\beta]}$. Thus $A^o_{[\alpha,\beta]}$ is a convex crisp set.

Conversely, since $\preceq_o$ is a total order, we can consider $c = \min_o\{A(x), A(z)\} \in L([0, 1])$. Then, $x, z \in A^o_c$. Since A^o_c is a convex crisp set, then $y \in A^o_c$ and so $\min_o\{A(x), A(z)\} \preceq_o A(y)$. □

The notion of level set connects IVFS with crisp sets. If we deal with the particular orders considered in Sect. 2, we obtain that Lo-convexity implies $Lex1$-convexity, $Lex2$-convexity and YX-convexity

In connection with the important property of the preservation of the convexity under intersections, we have obtained the following results.

Unfortunately, the Lo-intersection of two interval-valued fuzzy sets which are Lo-convex is not always Lo-convex, as we can see at the following counterexample.

Example 8 Let $X = \{x, y, z\}$ with $x < y < z$. If we consider the interval-valued fuzzy sets A and B defined as follows:

X	x	y	z
A	[0.1, 0.7]	[0.2, 0.8]	[0. 3,0.5]
B	[0.1, 0.7]	[0.4, 0.6]	[0. 3,0.5]
$A \cap_{Lo} B$	[0.1, 0.7]	[0.2, 0.6]	[0.3, 0.5]

Then A is Lo-convex, since $[0.1, 0.7] \preceq_{Lo} [0.2, 0.8]$ and B is Lo-convex since $[0.3, 0.5] \preceq_{Lo} [0.4, 0.6]$. However, $A \cap_{Lo} B$ is not Lo-convex since $[0.2, 0.6]$ is not related with $[0.1, 0.7]$ or $[0.3, 0.5]$ by means of the order relation $\preceq_{Lo}$.

For total orders we have obtained a general and positive result included in the following proposition.

Proposition 10 *Let X be an ordered space and let $\preceq_o$ a total order on $L([0, 1])$. If $A, B \in IVFS(X)$ are o-convex (resp. strictly o-convex), then $A \cap_o B$ is also o-convex (resp. strictly o-convex), whenever it is not empty.*

Proof Let x, y, z be three elements in X with $x < y < z$.

If $A(y) \preceq_o B(y)$, by Proposition 3 we have that $A \cap_o B(y) = A(y)$. Since A is o-convex (resp. strictly o-convex), $A(x) \preceq_o A(y)$ (resp. $A(x) \prec_o A(y)$) or $A(z) \preceq_o A(y)$ (resp. $A(z) \prec_o A(y)$). But by the definition of the intersection for this order we have that $A \cap_o B(x) \preceq_o A(x)$ and $A \cap_o B(z) \preceq_o B(z)$. By the transitivity, $A \cap_o B(x) \preceq_o A(y) = A \cap_o B(y)$ (resp. $A \cap_o B(x) \prec_o A(y) = A \cap_o B(y)$) or $A \cap_o B(z) \preceq_o A(y) = A \cap_o B(y)$ (resp. $A \cap_o B(z) \prec_o A(y) = A \cap_o B(y)$).

The case $B(y) \preceq_o A(y)$ is totally analogous. □

Finally, by applying Proposition 10 to the case of admissible order, and in particular the lexicographical orders and the Xu and Yager order, we obtain the following result.

Corollary 2 *If $\preceq_o$ is an admissible order, then o-convexity (resp. strictly o-convexity) is preserved under intersections.*

Finally, we will present some results about optimization that would be useful for the next section.

Theorem 2 *Let A be a convex IVFS over the ordered space X. Let $\preceq_o$ be an order on $L([0, 1])$. If $x^* \in supp(A) = \{x \in X : [0, 0] \prec_o A(x)\}$ is a strict local maximizer of $A(x)$, then it is also a global maximizer of $A(x)$ over $supp(A)$. The set of points at which $A(x)$ attains its global maximum over its support is a crisp convex set.*

Proof Suppose that $x^* \in supp(A)$ is a strict local maximizer. It means that there exists a neighborhood Y such that for all $x \in Y$, there is $A(x) \prec_o A(x^*)$. Let us suppose that there exists $x' \in supp(A)$, different from x^*, such that $A(x^*) \preceq_o A(x')$. By convexity, we have that $A(x') \preceq_o A(y)$ or $A(x^*) \preceq_o A(y)$, with $x'\langle y \langle x^*$ or $x^*\langle y \langle x'$. Then, if we take y close enough to x^*, that is, $y \in Y$ and $y \neq x^*$, that contradicts $A(y) \prec_o A(x^*)$. For the second part of the theorem, let us suppose that $[\alpha, \beta]$ is the element where $A(x)$ attains its maximum value. If we build the level set associated to $[\alpha, \beta]$ following (Huidobro et al. 2020) definition, it is a convex crisp set as A is a

convex IVFS. Huidobro et al. were able to proof the equivalence between the convexity of level sets and the convexity of IVFS. □

If we consider a strictly convex IVFS rather than a convex IVFS we are able to obtain better results. When we work with strict convexity the local maximizer is a global maximizer too, however, we need a strict local maximizer to assure that when considering just convexity (not strict). Another attractive point of considering a strictly convex IVFS is that when it achieves its maximum, it is unique.

Theorem 3 *Let A be a strictly convex IVFS over the ordered space X. Let $\preceq_o$ be an order on $L([0, 1])$.*

(i) *If $x^* \in supp(A)$ is a local maximizer of $A(x)$, then it is also a global maximizer.*
(ii) *$A(x)$ attains its maximum over $supp(A)$ at no more than one point.*

Proof (i) We can assume that there exists a neighborhood Y where x^* is a local maximizer. Let us suppose that there is x' a global maximizer such that $A(x^*) \prec A(x')$. By the strict convexity of A, we have that $A(x') \prec_o A(y)$ or $A(x^*) \prec_o A(y)$, with $x' < y < x^*$ or $x^* < y < x'$. If we choose y close enough to x^*, that is $y \in Y$, there is a contradiction.

(ii) Let us suppose that $x^*, x' \in supp(A)$ are global maximizers, that is $A(x) \prec_o A(x^*) = A(x')$ for all $x \in X$. By strictly convexity of A, we have that $A(x') \prec_o A(y)$ or $A(x^*) \prec_o A(y)$ with $x'\langle y\langle x^*$ or $x^* < y < x'$, and that contradicts the fact that there are two global maximizers.

□

5 Decision-making based on IVFSs

In this section we propose an application in decision-making problems. This proposal has been briefly introduced in Huidobro et al. (2021). In the literature, there are some theories about the use of fuzzy sets in decision-making. For example, Bellman and Zadeh (1970) showed that a decision could be seen as a group of goals and constraints with symmetry between them. This procedure enables us to deal with goals and constrains as if they were concepts that are joined in a symmetric way by "and" connective.

In fuzzy set theory, it is affirmed that we identify the membership degree of the elements to the set. Nevertheless, it usually occurs that the membership function is not known precisely. There exist situations where we are not certain on which is the precise value to indicate a fuzzy membership value, despite, we can resolve this problem by specifying an interval where the value is included.

We will use the approach of Bellman and Zadeh (1970), that is, if we contemplate the constraints and the goals as IVFS over the set of alternatives, X, thus the decision D would be the intersection of the interval-valued fuzzy constraints and goals.

In Yager and Basson (1975), a decision is constructed as the intersection of all the goals and the constraints. Keeping in mind this idea, we propose the following.

Definition 12 Let $X = \{x_1, , \ldots, x_n\}$ be the set of alternatives, $G_1, , \ldots G_p$ be the set of goals that can be expressed as IVFSs on the space of alternatives, and $C_1, \ldots, C_m$ be

the set of constraints that can also be expressed as IVFSs on the space of alternatives. Let $\preceq_o$ be an order on $L([0, 1])$. The goals and constraints then combine to form a decision D, which is an IVFS resulting from the intersection of the goals and the constrains. Thus, $D = G_1 \cap_o, \ldots \cap_o G_p \cap_o C_1 \cap_o \ldots \cap_o C_m$.

The meaning of $D(x)$ could signify the degree to that the alternative x fulfills the goals and constraints, for any $x \in X$. Once the decision is made, we have to decide the best alternative.

It is clear from this definition that D immediately depends on the selected order $\preceq_o$ in $L([0, 1])$ because the intersection is really an o-intersection. Thus, the decision D, which is the intersection of all the goals and constraints, would vary depending on the order we are considering.

Let us show an example based on Huidobro et al. (2021):

Example 9 A person has to choose to locate a new plant in one of three locations x_1, x_2 and x_3. He wants to select a location that minimizes real estate cost G, and is located near supplies, C_1. Let $X = \{x_1, x_2, x_3\}$. In this case, there is imprecision in the data, so IVFS would be more proper sets than FS. Let's suppose that the membership functions of the interval-valued fuzzy goal G is $\{\langle x_1, [0.25, 0.75]\rangle, \langle x_2, [0.65, 0.75]\rangle, \langle x_3, [0.45, 0.85]\rangle\}$ and the membership function of the interval-valued fuzzy constraint C_1 is $\{\langle x_1, [0.55, 0.65]\rangle, \langle x_2, [0.55, 0.95]\rangle, \langle x_3, [0.35, 0.65]\rangle\}$.

If we consider lexicographical order type 1, we emphasize the lower endpoint of the interval. Then the membership functions of the interval-valued fuzzy decision D_{lex1} is:

$$\{\langle x_1, [0.25, 0.75]\rangle, \langle x_2, [0.55, 0.95]\rangle, \langle x_3, [0.35, 0.65]\rangle\}$$

and the optimal decision would be x_2, since it is the alternative with a maximum value of D_{lex1} with respect to the lexicographical order type 1.

However, if we use lexicographical order type 2, then he membership functions of the interval-valued fuzzy decision D_{lex2} is:

$$\{\langle x_1, [0.55, 0.65]\rangle, \langle x_2, [0.65, 0.75]\rangle, \langle x_3, [0.35, 0.65]\rangle\}$$

and the optimal decision changes to x_3.

Following this easy example, the importance of a good selection of the order on $L([0, 1])$ is shown.

In the previous example, all the goals and constraints are interval-valued fuzzy sets over the same set of alternatives, however, there are situations where they are defined in a different set of alternatives. If we use the extension principle, we can avoid this situation.

Definition 13 (*Extension principle*) Let $\preceq_o$ be an order on $L([0, 1])$. Any given function $f : X \rightarrow Y$ induces two functions, $f : IVFS(X) \rightarrow IVFS(Y)$ and $f^{-1} : IVFS(Y) \rightarrow IVFS(X)$, which are defined by $[f(A)](y) = sup_{x|y=f(x)} A(x)$ for all $A \in IVFS(X)$, where sup denotes the supremum using the order $\preceq_o$ and $[f^{-1}(B)](x) = B(f(x))$ for all $B \in IVFS(Y)$.

With this procedure, when the interval-valued fuzzy constraints or goals are defined in different spaces, they can be mapped into the same space. When we have an n-ary function which maps $X_1 \times X_2 \times \cdots \times X_n$ to Y, we would assume that if $A \in IVFS(X_1 \times X_2 \times \cdots \times X_n)$, then $A(x_1, x_2, \ldots, x_n) = A(x_1) \cap_o A(x_2) \cap_o \cdots \cap_o A(x_n)$.

Let us show it by the following example based on Huidobro et al. (2021):

Example 10 Suppose the same conditions as in Example 9, but now there is another space Y meaning a set of former works developed by the potential financial directors, $Y = \{y_1, y_2, y_3, y_4\}$. We have some information about these former works: y_1 and y_2 were made by x_1, y_3 was supervised by x_2 and y_4 was produced by x_2 and x_3.

With this information we construct the following mappig:

$$f : Y \rightarrow X$$

defined by $f(y_1) = x_1$, $f(y_2) = x_1$, $f(y_3) = x_2$ and $f(y_4) = \{x_2, x_3\}$.

We also known a fuzzy constraint over Y that measures the impact of each one of works defined by: $C_2(Y) = \{\langle y_1, [0.45, 0.65]\rangle, \langle y_2, [0.75, 0.95]\rangle, \langle y_3, [0.78, 0.85]\rangle, \langle y_4, [0.65, 0.95]\rangle\}$. It is denoted as $C_2(Y)$ in order to point out that it is an interval-valued fuzzy set over the space Y. Now we should apply the extension principle to have all the goals and constraints as interval-valued fuzzy sets over the same space. To apply the extension principle we should first decide which order are we taking into account, in this case, we would use lexicographical order type 1. For x_1, $[f(C_2)](x_1) = sup_{y|y=f(x)} C_2(x) = sup_{y_1, y_2} C_2(x) = sup\{C_2(y_1), C_2(y_2)\} = [0.75, 0.95]$. Analogously, $[f(C_2)](x_2) = [0.78, 0.85]$ and $[f(C_2)](x_3) = [0.65, 0.95]$.

Consequently, $C_2(X) = \{\langle x_1, [0.75, 0.95]\rangle, \langle x_2, [0.78, 0.85]\rangle, \langle x_3, [0.65, 0.95]\rangle\}$.

Finally, the decision is $D = G \cap C_1 \cap C_2$, that is, the membership degrees for the different alternatives in D are:

$$\{\langle x_1, [0.25, 0.65]\rangle, \langle x_2, [0.55, 0.95]\rangle, \langle x_3, [0.35, 0.65]\rangle\}$$

Thus, the optimal decision is still x_2.

In some situations, any parameter in the decision could be conditional upon other space, Yager and Basson introduced the concept of fuzzy conditional set in Yager and Basson (1975). Considering these ideas, we propose the following definition:

Definition 14 An IVFS $B(y)$ in X is conditional on y if its membership function depends on y as a parameter. This dependence is denoted $B(x|y)$.

Then, if we are considering with two spaces, X and Y, and $y \in Y$, and there exists an interval-valued fuzzy set $B(y)$ on X, if we take $A \in IVFS(Y)$, thus A induces an IVFS B in X whose membership function is $B(x) = sup_y \min\{A(y), B(x|y)\}$.

Example 11 Suppose the conditions of Example 10. The company is forced to minimize the facility of employing workers. They would concentrate on the distance to the main office. Let $Y = \{Near(N), Med(M), Far(F)\}$. This constraint is given by the IVFS $C_3(Y) = \{\langle N, [0.85, 1]\rangle, \langle M, [0.45, 0.75]\rangle, \langle F, [0.15, 0.35]\rangle\}$. The relation between the alternatives and the proximity to the main office is given by the following conditioned IVFSs: $C_3(X|N) = \{\langle x_1, [0.75, 0.85]\rangle, \langle x_2, [0.55, 0.65]\rangle, \langle x_3, [0.35,$

$0.65]\rangle\}$, $C_3(X|M) = \{\langle x_1, [0.55, 0.65]\rangle, \langle x_2, [0.55, 0.75]\rangle, \langle x_3, [0.65, 0.95]\rangle\}$, and $C_3(X|F) = \{\langle x_1, [0.35, 0.75]\rangle, \langle x_2, [0.45, 0.65]\rangle, \langle x_3, [0.35, 0.75]\rangle\}$. Thus, we can construct the interval-valued fuzzy set facility of hiring workers:

For x_1, $C_3(x_1) = sup_y \min\{C_3(y), C_3(x_1|y)\} = sup\{\min\{C_3(N) = [0.85, 1], C_3(x_1|N) = [0.75, 0.85]\}, \min\{C_3(M) = [0.45, 0.75], C_3(x_1|M) = [0.55, 0.65]\}, \min\{C_3(F) = [0.15, 0.35], C_3(x_1|F) = [0.35, 0.75]\}\} = sup\{[0.75, 0.85], [0.45, 0.75], [0.15, 0.35]\} = [0.75, 0.85]$. We have to repeat the same procedure for x_2 and x_3. Thus, we obtain that the interval-valued fuzzy set $C_3(X)$ is given by $\{\langle x_1, [0.75, 0.85]\rangle, \langle x_2, [0.55, 0.65]\rangle, \langle x_3, [0.45, 0.75]\rangle\}$.

Finally, the decision is $D = G \cap C_1 \cap C_2 \cap C_3$, that is, the decision if the interval-valued fuzzy set D defined as:

$$\{\langle x_1, [0.25, 0.65]\rangle, \langle x_2, [0.55, 0.65]\rangle, \langle x_3, [0.35, 0.65]\rangle\}.$$

Thus, x_2 is again the optimal decision.

It is time to combine a decision-making problem with Theorems 2 and 3:

Corollary 3 *Let $\preceq_o$ be an order on $L([0, 1])$, let $G_1, \ldots, G_p$ be the interval-valued fuzzy goals, $C_1, \ldots, C_m$ the interval-valued fuzzy constraints, and $D = G_1 \cap, \ldots \cap G_p \cap C_1 \cap, \ldots \cap C_m$ be the resulting decision.*

- *If the interval-valued fuzzy goals and the interval-valued fuzzy constraints are convex IVFS, then the resulting decision D is a convex IVFS and the set of maximizing decisions of the IVFS D is a convex crisp set.*
- *If the interval-valued fuzzy goals and the interval-valued fuzzy constraints are strictly convex IVFS, then the resulting decision D is a strictly convex IVFS and the set of maximizing decisions of D is a singleton or an empty set.*

Let us summarize the decision-making problem of Example 11 in the following example based on Huidobro et al. (2021):

Example 12 In the previous examples we consider one interval-valued fuzzy goal $G = \{\langle x_1, [0.25, 0.75]\rangle, \langle x_2, [0.65, 0.75]\rangle, \langle x_3, [0.45, 0.85]\rangle\}$ and three interval-valued fuzzy constraints $C_1 = \{\langle x_1, [0.55, 0.65]\rangle, \langle x_2, [0.55, 0.95]\rangle, \langle x_3, [0.35, 0.65]\rangle\}$, $C_2 = \{\langle x_1, [0.75, 0.95]\rangle, \langle x_2, [0.78, 0.85]\rangle, \langle x_3, [0.65, 0.95]\rangle\}$ and $C_3 = \{\langle x_1, [0.75, 0.85]\rangle, \langle x_2, [0.55, 0.65]\rangle, \langle x_3, [0.45, 0.75]\rangle\}$. If we suppose $x_1 < x_2 < x_3$, it is clear that G, C_1, C_2 and C_3 are strictly convex IVFS with respect to the lexicographical order type 1, so the decision D is also a convex IVFS w.r.t. the same order. It is easy to check, since $D = \{\langle x_1, [0.2, 0.7]\rangle, \langle x_2, [0.5, 0.6]\rangle, \langle x_3, [0.3, 0.6]\rangle\}$. We can apply the previous result to assert that x_2 is a global maximizer.

As changing the order could be also interesting, in the following example we show what happens if we use lexicographical order type 2.

Example 13 Using the same IVFS for the goal and contraints from the previous example, the decision-making problem is $G = \{\langle x_1, [0.25, 0.75]\rangle, \langle x_2, [0.65, 0.75]\rangle, \langle x_3, [0.45, 0.85]\rangle\}$, $C_1 = \{\langle x_1, [0.55, 0.65]\rangle, \langle x_2, [0.55, 0.95]\rangle, \langle x_3, [0.35, 0.65]\rangle\}$, $C_2 = \{\langle x_1, [0.75, 0.95]\rangle, \langle x_2, [0.65, 0.95]\rangle, \langle x_3, [0.65, 0.95]\rangle\}$ and $C_3 = \{\langle x_1, [0.75, 0.85]\rangle$,

$\langle x_2, [0.45, 0.75] \rangle, \langle x_3, [0.45, 0.75] \rangle \}$. It should be noticed that there are changes in the constrains C_2 and C_3 because we used lexicographical order type 2 and it affects to the supremum and the minimum. Moreover, the constraints C_2 and C_3 are convex IVFS while G and C_1 are strctly convex IVFS. We can also see that D is a convex IVFS, as $D = \{\langle x_1, [0.55, 0.65] \rangle, \langle x_2, [0.45, 0.75] \rangle, \langle x_3, [0.35, 0.65] \rangle \}$. Thus, D is not only convex but strictly convex, so we can assure that x_1 is the unique optimal decision.

6 Concluding remarks

In this paper, we present a definition of convexity for IVFS based on an order relation among intervals, which fulfills natural properties: convexity is preserved under intersections and the cutworthy property. The chosen order plays a significant part since the order relation change with it, so several definitions of intersection appear. Although the usual definition of an intersection found in the literature coincides with the one generated by the lattice order, convexity is not preserved, so it is not surprising that not all of the orders between intervals are appropriate for defining the intersection. However, admissible orders seem to be more suitable for this purpose. Similarly, admissible orders work well with the union of IVFS. To continue with convexity, we suggested a proper definition of level sets and studied some interesting properties about them. After that, we adapted the decomposition theorem of fuzzy sets to IVFS and applied it to characterize an IVFS through its level sets. Finally, we introduce a method to use interval-valued fuzzy sets and convexity to optimization or decision-making problems. It should be pointed out that, when designating membership functions to the sets, the subjectivity of IVFSs may help to define convex interval-valued goals and constraints. We were also allowed to prove that a local maximizer could be easily a global maximizer. It should be noticed that the chosen order on $L([0, 1])$ is really connected to the optimal decision.

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