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IFAC PapersOnLine 58-9 (2024) 1–6

Algebraic tools in autotuning design

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Abstract: Autotuners represent a combination of a relay feedback identification test and some control design method. In this contribution, models with up to three parameters are estimated by means of a single asymmetrical relay experiment. Then a stable low order transfer function with a time delay term is identified by a relay experiment. Autotuning principles then combine asymmetrical relay feedback tests with a control synthesis. Two algebraic control syntheses then are presented in this paper. The first one is based on the ring of proper and stable rational functions R_{PS}. The second one utilizes a special ring R_{MS}, a set of RQ-meromorphic functions. In both cases, controller parameters are derived through a general solution of a linear Diophantine equation in the appropriate ring. A final controller can be tuned by a scalar real parameter $m>0$. The presented philosophy covers a generalization of PID controllers and the Smith-like control structure. This contribution deals with the implementation of proposed autotuners and presents some illustrative examples.

Keywords: Diophantine equation, Rings, Relay experiment, Autotuning, Pole-placement

1. INTRODUCTION

The paper presents two possibilities how to design controllers for simple delayed systems. There surely exist several approaches of control design methods, nowadays, three main groups dominate. The first group contains approaches based on the Smith predictor structure, or more precisely its modifications, see e.g. Majhi and Atherton (1998), Majhi (2007). These methods assume model of the controlled system in feedback loops in the sense of IMC (Internal Model Controllers). The second group consists of predictive based approaches, mainly using state-space description. The third approach is based on algebraic tools and methods and its development can be traced in Vidyasagar (1985), Kučera (1993), Prokop and Corriou (1997), Zítek and Kučera (2003), etc. Methods from the third group are described in the paper. The first design in the paper utilizes a ring of stable and proper meromorphic functions R_{MS} omitting any approximation which was developed especially for delay systems, Zítek and Kučera (2003).The second one is based on the ring of stable and proper rational function R_{PS}, see Vidyasagar (1985), Kučera (1993), Prokop and Corriou (1997). Many industrial processes can be modeled with stable systems with a delay time term. The contribution brings controller design for first order (stable) delayed (FODS) and second order (SODS) models. For both systems, the control syntheses are performed that are applied in R_{MS} and in R_{PS}, as a special case with neglecting of the time delay term. Some of the developed controllers are no more in PI/PID structure. Subsequently, they are utilized in an autotuning scheme and then compared in control of a high order system with a time delay term.

2. SYSTEM DESCRIPTION

This contribution deals with two simplest SISO linear dynamic systems with a delay term. The first model of the first order (stable) plus dead time (FOPDT) is supposed in the form:

$$G(s)=\frac{K}{Ts+1}\cdot e^{-\tau s} \tag{1}$$

The second order model plus dead time (SOPDT) is assumed in the form:

$$G(s)=\frac{K}{(Ts+1)^2}\cdot e^{-\tau s} \tag{2}$$

2.2 Models in R_{PS}

The traditional engineering design approach of PID like controllers was performed either in the frequency domain or in polynomial representation, see e. g. Åström and Hägglund, (1995). Nevertheless, the fractional approach developed in (Vidyasagar, 1985; Kučera, 1993) and analyzed in (Prokop and Corriou, 1997; Prokop *et al.*, 2005, 2010) enables a deeper insight into control tuning and a more elegant expression of all suitable controllers.

It is well known that a set of polynomials is a ring. However, there are other rings which can be used for the system description. The fractional approach supposes that transfer functions of continuous-time linear causal systems in R_{PS} can be expressed as a ratio of two elements:

10.1016/j.ifacol.2024.07.362

$$G(s)=\frac{b(s)}{a(s)}=\frac{\dfrac{b(s)}{(s+m)^n}}{\dfrac{a(s)}{(s+m)^n}}=\frac{B(s)}{A(s)} \tag{3}$$

2.3 Models in R_{MS}

An element of this ring R_{MS} is a ratio of two retarded quasipolynomials $y(s)/x(s)$. A retarded quasipolynomial $x(s)$ of degree n means

$$x(s)=s^n+\sum_{i=0}^{n-1}\sum_{j=1}^{h}x_{ij}s^i\exp\left(-\vartheta_{ij}s\right),\ \vartheta_{ij}\geq 0 \tag{4}$$

where *retarded* refers to the fact that the highest s-power is not affected by exponentials. A more general notion called neutral quasipolynomials also can be used in this sense, see Pekař et al. (2010). Quasipolynomial (4) is stable when it owns no finite zero s_0 such that $\mathrm{Re}\{s_0\}\geq 0$. For stability tests, see e.g. in Vyhlídal and Zítek (2001), Zítek and Kučera (2003), Pekař (2012), Prokop et al. (2016).

Transfer function with time delay is considered as a fraction of two quasipolynomials and the denominator $m(s)$ is a stable quasipolynomial. As an example, a FODS can be expressed by

$$\tilde{G}(s)=\frac{\dfrac{Ke^{-\tau s}}{s+m_0e^{-\theta s}}}{\dfrac{Ts+1}{s+m_0e^{-\theta s}}}=\frac{\dfrac{b(s)}{m(s)}}{\dfrac{a(s)}{m(s)}}=\frac{B(s)}{A(s)},\ m_0>0 \tag{5}$$

It is naturally possible to use factorization (5) with $\Theta=0$ or $\Theta=\tau$. For stable processes the option $\Theta=0$ is satisfactory; however, for unstable processes should be taken $\Theta\neq 0$. The choice $\Theta=0$ gives a traditional FOPDT system, see Yu (1999). The ratio after the second equal sign in (5) represents a generalization in R_{MS} description with *a(s), b(s), m(s)* quasipolynomials.

3. CONTROL DESIGN

The control loop is considered as a simple feedback system (1DOF) with a controller $Q(s)/P(s)$ and a controlled plant $B(s)/A(s)$, depicted in Fig.1. The second possibility is to assume control loop in the 2DOF structure depicted in Fig. 2.

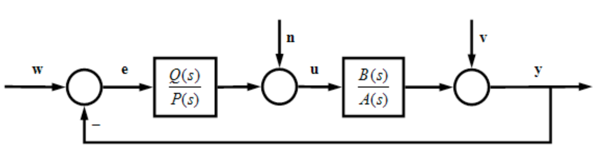

Figure 1. One degree of freedom (1DOF) control loop

The aim of the control synthesis is to stabilize a feedback control system, obtain asymptotic tracking and attenuate load disturbance.

All feedback stabilizing controllers for the feedback system are given by a general solution of the Diophantine equation (A, B coprime):

$$AP+BQ=1 \tag{6}$$

which can be expressed with Z free in R_{PS}:

$$\frac{Q}{P}=\frac{Q_0-AZ}{P_0+BZ},\quad P_0+BZ\neq 0 \tag{7}$$

In contrast to the polynomial design, all controllers are proper and can be utilized.

The Diophantine equation for designing the feedforward part in the structure 2DOF controller is:

$$F_wS+BR=1 \tag{8}$$

with parametric solution:

$$\begin{aligned}R&=R_0-F_WZ\\ S&=S_0+BZ\end{aligned} \tag{9}$$

In the case of 1DOF structure, asymptotic tracking is ensured by the divisibility of the denominator P by the denominator of the reference $w=G_w/F_w$. Asymptotic tracking in the case of 2DOF structure is achieved by the solution of the second Diophantine equation (8). The most frequent case for w is a stepwise reference signal with the denominator in the form $F_w=\dfrac{s}{s+m};\quad m>0$.

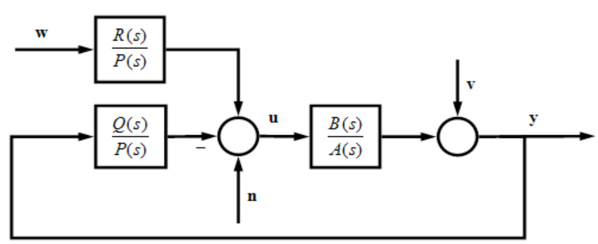

Figure 2. Two-degree of freedom (2DOF) control loop

3.1 PI and PID controllers in R_{PS}

Simplest cases for SISO systems (1) is a first order controlled system (plus time delay FOPDT). Diophantine equation (6) for the first order systems (1) without a time delay term can be easily transformed into polynomial equation:

$$(Ts+1)p_0+Kq_0=s+m \tag{10}$$

with general solution:

$$\begin{aligned}P&=\frac{1}{T}+\frac{K}{s+m}\cdot Z\\ Q&=\frac{Tm-1}{TK}-\frac{Ts+1}{s+m}\cdot Z\end{aligned} \tag{11}$$

where Z is free in the ring R_{PS}. Asymptotic tracking is achieved by the choice $Z = -\frac{m}{TK}$ and the resulting PI controller is in the form:

$$C(s) = \frac{Q}{P} = \frac{q_1 s + q_0}{s} \tag{12}$$

where parameters q_1 a q_0 are given by:

$$q_1 = \frac{2Tm - 1}{K} \qquad q_0 = \frac{Tm^2}{K} \tag{13}$$

For the SOPDT the design equation takes the form:

$$(Ts+1)^2 \cdot (p_1 s + p_0) + K \cdot (q_1 s + q_0) = (s+m)^3 \tag{14}$$

and after similar manipulations the resulting PID controller gives the transfer function:

$$C(s) = \frac{Q}{P} = \frac{q_2 s^2 + q_1 s + q_0}{s(s + p_0)} \tag{15}$$

For both systems FOPDT and SOPDT the scalar parameter $m>0$ seems to be a suitable „tuning knob" influencing control behavior as well as robustness properties of the closed loop system. The derivation for SOPDT can be found in Prokop et al. (2015a, 2015b).

3.2 Controllers in R_{MS}

The first step of the control design is to stabilize the system by a proper feedback loop. It can be formulated in an elegant way in R_{MS} by the Diophantine equation

$$A(s)P_0(s) + B(s)Q_0(s) = 1 \tag{16}$$

where $P_0(s)$ a $Q_0(s)$ is a particular solution. Then, all stabilizing controllers can be expressed in a parametric form by

$$\frac{Q(s)}{P(s)} = \frac{Q_0(s) + A(s)Z(s)}{P_0(s) - B(s)Z(s)}, \quad P_0(s) - B(s)Z(s) \neq 0 \tag{17}$$

where $Z(s)$ is an arbitrary element of R_{MS}. The special choice of this element can ensure additional control conditions. Details and proofs can be found e.g. in Zítek and Kučera (2003). Let the reference and load disturbance be expressed by

$$\Omega(\sigma) = H_\Omega(\sigma)/\Phi_\Omega(\sigma), \;\; N(\sigma) = H_\Delta(\sigma)/\Phi_\Delta(\sigma) \tag{18}$$

respectively, then conditions for asymptotic tracking and disturbance attenuation result from expression for $E(s)$ which reads

$$E(s) = \frac{A(s)P(s)}{A(s)P(s) + B(s)Q(s)} W(s) - \frac{B(s)P(s)}{A(s)P(s) + B(s)Q(s)} N(s) \tag{19}$$

It is required that $E(s)$ must belong to R_{MS}. In other words, it is demanded that both $F_w(s)$ and $F_N(s)$ divides $P(s)$. Details about divisibility in R_{MS} and R_{PS} can be found, e.g. in Zítek and Kučera (2003), Pekař and Prokop (2008). The most frequent case is that both signals $w(t)$ and $n(t)$ can be considered as step functions. Thus, for the case of R_{PS} ring, it is equivalent to reach the absolute term of $P(s)$ equal to zero. The last condition is not possible to reach in R_{MS}, due to $B(s)$ and/or $P_0(s)$ containing delay term $e^{-\tau s}$. For this case, the following expression for the absolute term in $P(s)$ is demanded $\lambda\left(1 - e^{-\tau s}\right)$ where λ is a selected real parameter, usually $\lambda = m_0^r$, where r is the order of the controlled system. This condition says that the controller $G_R(s)=Q(s)/P(s)$ has integral behavior for $s \to 0$. It can be assured by proper choice of $Z(s)$ in (7). If $w(t)$ or $n(t)$ is another function, divisibility conditions can be more complex.

The controller design of a stable FODS solve (6) by the choice $Q_0 = 1$ yielding

$$P_0(s) = \frac{s + m_0 - K\exp(-\tau s)}{Ts + \exp(-\vartheta s)} \tag{20}$$

Now parameterize the solution according to (7) to obtain controllers asymptotically rejecting the disturbance

$$P(s) = \frac{s + m_0 - K\exp(-\tau s)}{Ts + \exp(-\vartheta s)} - \frac{K\exp(-\tau s)}{s + m_0} Z(s) \tag{21}$$

The numerator of $P(s)$ has to have at least one zero root. Moreover, it is appropriate to have $P(s)$ in a simple form, which is fulfilled e.g. when

$$Z(s) = \left(\frac{m_0}{K} - 1\right) \frac{s + m_0}{Ts + \exp(-\vartheta s)} \tag{22}$$

providing

$$P(s) = \frac{s + m_0\left(1 - \exp(-\tau s)\right)}{Ts + \exp(-\vartheta s)}, \; Q = \frac{m_0}{K} \tag{23}$$

Thus, final controller's structure is the following

$$G_R(s) = \frac{m_0\left[Ts + \exp(-\vartheta s)\right]}{K\left[s + m_0\left(1 - \exp(-\tau s)\right)\right]} \tag{24}$$

The control design for according to Fig.1 (SOPDT) consists in (16) which takes the form

$$(Ts+1)^2 P_0(s) + Ke^{-\tau s} Q_0(s) = (s + m_0)^2 \tag{25}$$

By the choice $Q_0(s) = 1$, the solution of (14) is obtained as

$$P_0(s) = \frac{(s + m_0)^2 - Ke^{-\tau s}}{(Ts+1)^2} \tag{26}$$

and the general solution of (4) is given by

$$\frac{Q(s)}{P(s)}=\frac{1+\frac{(Ts+1)^2}{(s+m_0)^2}Z(s)}{\frac{(s+m_0)^2-Ke^{-\tau s}}{(s+a_0)^2}-\frac{Ke^{-\tau s}}{(s+m_0)^2}Z(s)} \tag{27}$$

The choice for *Z(s)*

$$Z(s)=\frac{\kappa(s+m_0)^2}{(Ts+1)^2} \tag{28}$$

gives *P(s)* in a very simple form and κ is a real free parameter. By the choice $\kappa = (m_0^2/K) - 1$, expression (15) is satisfied, and

$$P(s)=\frac{s^2+2m_0s+m_0^2\left(1-e^{-\tau s}\right)}{(Ts+1)^2};\ Q(s)=\frac{m_0^2}{K} \tag{29}$$

Thus, the final form of the controller $G_R(s)$ is then

$$G_R(s)=\frac{m_0^2}{K}\frac{(Ts+1)^2}{s^2+2m_0s+m_0^2\left(1-e^{-\tau s}\right)} \tag{30}$$

where $m_0 > 0$ is a real positive tuning parameter which can be tuned, one of them as an "equalization principle" (Prokop et al. 2010a, b). Since that controller (19) is in the quasipolynomial form, denominator in (19) has infinite number of poles. The construction of this controller is more complex than usual PI or PID controllers. However, modern PLC systems facilitate using advanced functions of the so-called anisochronic controller. A case when $\tau = 0$ gives PI and PID controllers.

4. RELAY IDENTIFICATION

An auto-tuning procedure consists of a process identification experiment plus a controller design method. The traditional method was proposed by Åström and Hägglund (1984, 1995) based on a symmetrical relay feedback test when a relay of magnitude h_r is inserted in the feedback loop. A wide class of autotuning principles can be found in Yu (1999). In this paper, an asymmetrical relay with hysteresis is used. This relay enables to estimate transfer function parameters as well as a time delay term. The feedback relay experiment is familiar known (e.g. Åström and Hägglund 1984, Yu 1999). Asymmetrical relay oscillations can be seen in Fig. 3.

The process gain for (1), (2), (5) can be computed by the relation (Prokop et al, 2010a, 2010b)

$$K=\frac{\int_0^{iT_y} y(t)dt}{\int_0^{iT_y} u(t)dt};\quad i=1,2,3,.. \tag{31}$$

The time constant and time delay terms for FOPDT are given by:

$$T=\frac{T_y}{2\pi}\cdot\sqrt{\frac{16\cdot K^2\cdot u_0^2}{\pi^2\cdot a_y^2}-1}$$
$$\tau=\frac{T_y}{2\pi}\cdot\left[\pi-arctg\frac{2\pi T}{T_y}-arctg\frac{\varepsilon}{\sqrt{a_y^2-\varepsilon^2}}\right] \tag{32}$$

Relations for the second order model are

$$T=\frac{T_y}{2\pi}\cdot\sqrt{\frac{4\cdot K\cdot u_0}{\pi\cdot a_y}-1}$$
$$\tau=\frac{T_y}{2\pi}\cdot\left[\pi-2arctg\frac{2\pi T}{T_y}-arctg\frac{\varepsilon}{\sqrt{a_y^2-\varepsilon^2}}\right] \tag{33}$$

where a_y and T_y are depicted in Fig. 3 and ε is the hysteresis.

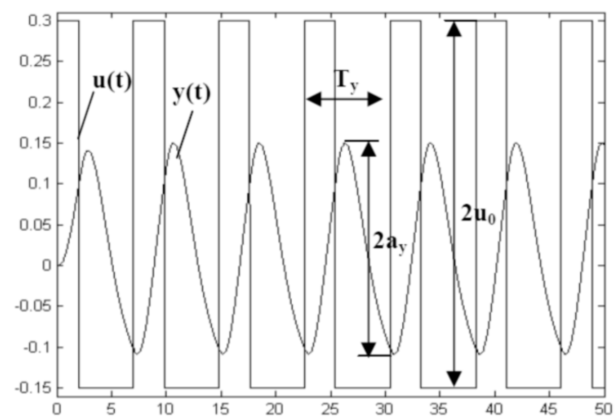

Figure 3. Asymmetrical relay oscillations

5. SIMULATION AND ANALYSIS

Consider a stable system with time delay governed by the transfer function

$$G(s)=\frac{2}{(s+1)^8}\cdot e^{-3s} \tag{34}$$

Transfer functions (34) represent a wide class of frequent stable industrial processes. The approximated system was identified by a relay experiment mentioned above. The first order approximation (34) gives

$$\tilde{G}(s)=\frac{2}{3.45s+1}\cdot e^{-8.20s} \tag{35}$$

The quantity of the normalized time delay $L=\tau/(T+\tau)=0.70$ for (35) indicates the difficulty of controlling. The second order approximation according to (33) is

$$\tilde{\tilde{G}}(s)=\frac{2}{(2.21s+1)^2}\cdot e^{-6.70s} \tag{36}$$

A simpler class of controllers was derived in R_{PS} representation (13), (15) with neglecting of time delay and the first order controller takes the form (for an appropriate choice of the tuning parameter, see Prokop et al., 2011)

$$G_{R1}(s)=\frac{0.02s+0.04}{s} \qquad m_0=0.15 \tag{37}$$

The second order R_{PS} controller gives

$$G_{R2}(s)=\frac{0.57s^2+0.40s+0.07}{3.40s^2+s} \qquad m_0=0.30 \tag{38}$$

A second class of anisochronic controllers was derived according to methodology (20) - (30). The first order R_{MS} controller for the given $m_0 > 0$ is then

$$G_{R11}(s)=\frac{0.16}{2}\cdot\frac{3.45s+1}{s+0.16(1-e^{-8.20s})} \qquad m_0=0.16 \tag{39}$$

and the second order R_{MS} controller takes the form

$$G_{R22}(s)=\frac{0.35^2}{2}\cdot\frac{(2.21s+1)^2}{s^2+0.7s+0.35^2(1-e^{-6.70s})} \qquad m_0=0.35 \tag{40}$$

First and second order time responses for the original plant (34) are shown in Fig. 4. Open loop ($G_R.G$) Nyquist plots of original plant (34) and anisochronic controller for various values of the tuning parameter are shown in Fig. 5. The gain and phase stability margins can be concluded from the plots.

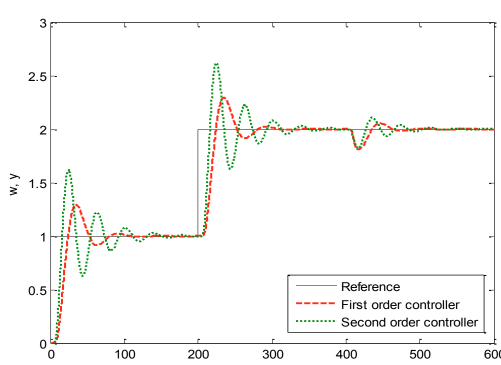

Figure 4. R_{PS} control responses of (34) and (37), (38)

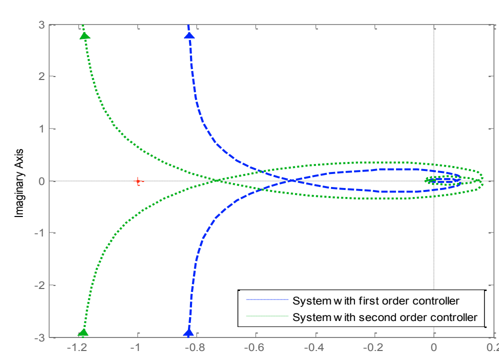

Figure 5. R_{PS} Nyquist plots of (34) and (37), (38)

Control responses for the R_{MS} first order controller (39) is shown in Fig. 6. The response for the second order controller is similar.

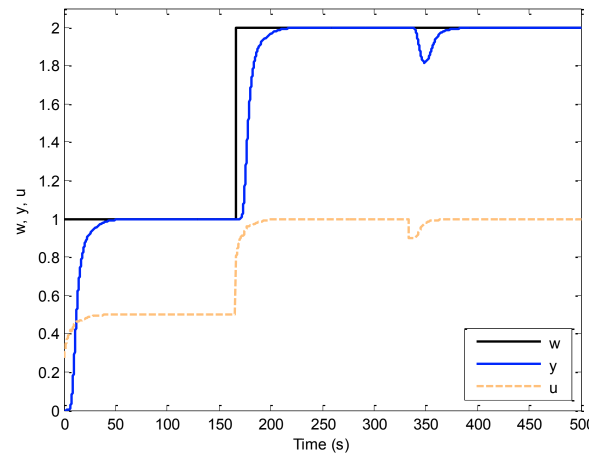

Figure 6. R_{MS} first order control response of (34) and (39)

Fig. 7 and Fig. 8 demonstrate open loop Nyquist plots and stability margins for various tuning parameters. Further robust aspects for delayed systems are studied e.g. *in* Pekař et al. (2010), Pekař (2012).

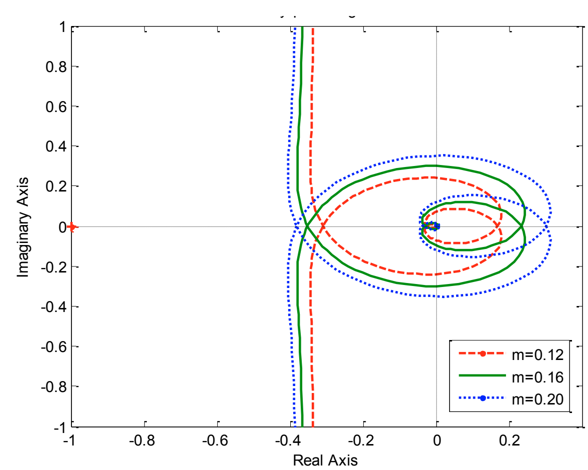

Figure 7. R_{MS} first order Nyquist plots of (34) and (39)

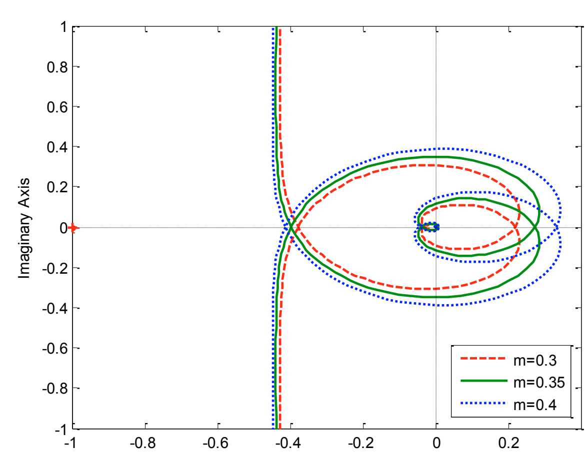

Figure 8. R_{MS} second order Nyquist plots of (34) and (40)

6. CONCLUSIONS

The contribution illustrates a utilization of algebraic tools in the field of autotuners. The control synthesis is performed through a solution of a Diophantine equation in two rings. The first one is the ring of proper and stable functions R_{PS}, the second one is the ring of RQ-meromorphic functions R_{MS}. This approach utilizes quasipolynomials and yields a class of Smith predictor like controllers. A special case, a rational R_{PS} function approach generates a class of generalized PID controllers. Both design methodologies bring a scalar tuning parameter $m_0 > 0$ that can be adjusted by various strategies. All derived algorithms were utilized as autotuning schemes with combination of relay feedback estimation. Transfer function parameters are estimated from asymmetric limit cycle data by a relay with hysteresis. The estimated transfer function is stable and time delay terms are accepted. The methodology is illustrated by the example of higher order and time delay. Better control responses as well as stability margins were achieved by anisochronic controllers while the R_{PS} function approach gives a simpler controller structure.

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