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RESEARCH ARTICLE

A purely algebraic proof of the omega-reducibility of pseudovarieties representing low half levels of concatenation hierarchies

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Abstract
We are concerned with the ω-reducibility of pseudovarieties of ordered monoids representing half levels of concatenation hierarchies. In the author's paper (Int. J. Algebra Comput. **64**(01), 87–135, 2024), the ω-reducibility of pseudovarieties representing levels 1/2 and 3/2 of concatenation hierarchies with a locally finite basic pseudovariety has been proven, using results of the paper by Place (Log. Methods Comput. Sci. **14**(4:16), 1–58, 2018) on so called covering of corresponding sets of regular languages. In this paper, we prove the same results on the ω-reducibility, not using the results of the mentioned paper by Place, although still inspired by their proofs. This new method of the proofs of the ω-reducibility prepares us to their potential extension to higher half levels of concatenation hierarchies. The process of a gradual generalization is initiated in this paper.

Keywords Concatenation hierarchy · Omega-reducibility · Ordered monoid · Pseudovariety

1 Introduction

This paper deals with an algebraic counterpart of concatenation hierarchies of regular languages. Originally, an instance of a concatenation hierarchy – the dot-depth hierarchy – was defined [14]. The dot-depth hierarchy is a hierarchy of classes of star-free languages. A star-free language is a language that can be created from letters by a finite number of applications of the operations of concatenation, union, and complementa-

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tion. Since the class of all regular languages is also closed under aforementioned operations, star-free languages form a subclass of the class of all regular languages. In the dot-depth hierarchy, the level which a given language belongs to, corresponds to a number of necessary alternations between the Boolean operations and the operation of concatenation to build the language from letters.

Later, this definition was generalized to all classes of regular languages [19]. A *concatenation hierarchy* of regular languages consists of full and half levels. Level zero is an arbitrary class of regular languages having certain "nice" closure properties. Then every half level $n + 1/2$ is created from the preceding full level n by taking the closure under the operations of union and concatenation and every full level $n + 1$ is created from the preceding half level $n + 1/2$ by taking the closure under the Boolean operations (i.e., union and complementation).

Due to Eilenberg-type correspondence [15, 18], levels of a concatenation hierarchy can be represented by corresponding pseudovarieties of finite semigroups/monoids. In this paper, we work with pseudovarieties of finite *monoids*. Using this correspondence, problems on concatenation hierarchies can be transferred into the theory of finite monoids. This is exactly what we use in this paper. We study concatenation hierarchies just from the viewpoint of the corresponding pseudovarieties of finite monoids. More precisely, we are interested just in *half* levels of concatenation hierarchies in this paper, which correspond to pseudovarieties of finite *ordered* monoids.

A *pseudovariety of ordered monoids* is an analogue of a variety of ordered monoids. The only difference is that a pseudovariety consists of *finite* ordered monoids. It is a nonempty class of *finite* ordered monoids that is closed under ordered submonoids, *finite* direct products and homomorphic images.

In this paper, we study a property of pseudovarieties of ordered monoids called *ω-reducibility*. We use this notion in the meaning established in [4]. The *ω-reducibility* of a pseudovariety V can be defined using the notion of V-*pointlike sets*, which have appeared to be an important tool in solving membership problems for levels of concatenation hierarchies [25, 28–30, 37].

Before saying more about *pointlike sets* and the *ω-reducibility*, let us mention a way of describing pseudovarieties, analogous to a description of *varieties* of ordered monoids by *inequalities* formed of pairs of *words* [10]. In our case of *finite* ordered monoids, *pseudovarieties* can be described by *pseudoinequalities* formed of pairs of *pseudowords* [17, 22]. Given a generating alphabet A, a *word* over A is an element of the *free monoid* over A, while a pseudoword over A is an element of the *free profinite monoid* over A.

Now we return to pointlike sets. Let M be a finite A-generated monoid. A V-*pointlike* set of M is an "imprint" of a set X of *pseudowords* over A, i.e., an "imprint" of a subset $X \subseteq \overline{\Omega}_A\mathsf{M}$ of the free profinite monoid $\overline{\Omega}_A\mathsf{M}$, through a continuous homomorphism $\varphi\colon \overline{\Omega}_A\mathsf{M} \to M$, where the set X "behaves like a point" with respect to V. More precisely, a set $\{s_1, \ldots, s_n\} \subseteq M$ is called V-*pointlike* if there exists a set of pseudowords $\{x_1, \ldots, x_n\} \subseteq \overline{\Omega}_A\mathsf{M}$ and a continuous homomorphism $\varphi\colon \overline{\Omega}_A\mathsf{M} \to M$ such that $\varphi(x_i) = s_i$ for $i = 1, \ldots, n$ and the pseudoidentities $x_1 = \cdots = x_n$ are valid in V. We will work with an analogue of pointlike sets for pseudovarieties of *ordered* monoids.

The free profinite monoid is uncountable, therefore it is not easy to work with it. It is much more convenient to work with some countable submonoid of the free profinite monoid. In this paper, we are concerned with replacing pseudoinequalities formed of arbitrary pseudowords by *ω-inequalities* formed of *ω-words*. An *ω-word* is a pseudoword that can be created from letters using a finite number of applications of the operations of concatenation and ω-power. For example, $(ab^{\omega}bba)^{\omega}aaa^{\omega}b$ is an ω-word over the alphabet $A = \{a, b\}$. The set of all ω-words over a fixed finite alphabet A forms a countable submonoid $\Omega_A^{\omega}\mathsf{M}$ of the free profinite monoid $\overline{\Omega}_A\mathsf{M}$ over A.

Finally we get to the notion of *ω-reducibility*. A pseudovariety of monoids V is called *ω-reducible* if, for every finite monoid M and for every two-element V-pointlike set $\{s_1, s_2\} \subseteq M$, the relevant pseudowords $x_1 \in \varphi^{-1}(s_1)$, $x_2 \in \varphi^{-1}(s_2)$ from the definition of a V-pointlike set can be chosen from the monoid $\Omega_A^{\omega}\mathsf{M}$ of ω-words.

In the case when V is a pseudovariety of *ordered* monoids, we consider an analogue of a two-element V-pointlike set – a so called V-*inequality-like pair*, which is an "imprint", through a continuous homomorphism $\varphi\colon \overline{\Omega}_A\mathsf{M} \to M$, of a pseudo*inequality* over A valid in V. Then a pseudovariety of *ordered* monoids V is called *ω-reducible* if, for every finite monoid M and for every V-*inequality-like pair* $(s_1, s_2) \in M \times M$, the relevant pseudowords $x_1 \in \varphi^{-1}(s_1)$, $x_2 \in \varphi^{-1}(s_2)$ from the definition of a V-inequality-like pair can be chosen from the monoid $\Omega_A^{\omega}\mathsf{M}$ of ω-words.

The reason for studying the ω-reducibility of pseudovarieties representing levels of concatenation hierarchies is the following. A pseudovariety of ordered monoids $\mathsf{V}_{n+1/2}$ representing half level $n + 1/2$ of a concatenation hierarchy, where $n \in \mathbb{N}$, can be characterized by the system of pseudoinequalities of the form

$$u^{\omega+1} \leq u^{\omega}vu^{\omega}, \tag{1}$$

where u, v are arbitrary pseudowords such that the pseudoinequality $v \leq u$ is valid in the pseudovariety $\mathsf{V}_{n-1/2}$ representing the preceding half level $n - 1/2$ [2]. Now if the pseudovariety $\mathsf{V}_{n-1/2}$ is ω-reducible, the pseudowords u, v in Inequality (1) can be replaced by ω-words. This means that if the pseudovariety $\mathsf{V}_{n-1/2}$ is ω-reducible, then the pseudovariety $\mathsf{V}_{n+1/2}$ is definable by ω-inequalities. The ω-inequalities can be considered as a sort of "simple" pseudoinequalities. Therefore, the ω-reducibility of a pseudovariety representing a half level of a concatenation hierarchy gives us a "simple" description of the next half level.

Moreover, by results of [6], if a pseudovariety V has the following two properties, then the *membership problem* for V is decidable, i.e., it is decidable whether a given finite (ordered) monoid belongs to V:

1. the pseudovariety V is definable by ω-inequalities,
2. it is decidable whether a given ω-inequality is valid in V.

By [5], Item 2 holds for all levels of the much studied concatenation hierarchy with the trivial level 0 – the *Straubing–Thérien hierarchy*. Therefore, by the preceding, the ω-reducibility of a pseudovariety representing a half level of the Straubing–Thérien hierarchy directly implies the decidability of the membership problem for the next half level.

It is important to notify the reader at this point that this paper does not give new results concerning the membership problem for levels of a concatenation hierarchy, since we work just with pseudovarieties representing the levels of concatenation hierarchies for which the membership problem has been already solved. However, the mentioned way of solving the membership problem using the ω-reducibility can motivate us to study the ω-reducibility of also other (in particular, higher) levels of concatenation hierarchies.

In the author's previous paper on the ω-reducibility [37], it has been proven that, for every concatenation hierarchy with a locally finite basic pseudovariety V_0, the pseudovarieties $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$ are ω-reducible. By the preceding, this implies that the pseudovarieties $\mathsf{V}_{3/2}$ and $\mathsf{V}_{5/2}$ are definable by ω-inequalities. Note that the membership problem for $\mathsf{V}_{3/2}$ and $\mathsf{V}_{5/2}$ had been solved before [25].

In [37], the ω-reducibility of the pseudovarieties $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$ was proven using results of [25] on so called *covering* of regular languages for the corresponding levels $1/2$ and $3/2$ of a given concatenation hierarchy. To use the results of [25], it was necessary to translate them into terms of pseudovarieties, in particular, to describe precisely a connection between the covering of regular languages and generalized pointlike sets for pseudovarieties of ordered monoids. It turned out that the algorithms invented in [25], used there to solve the covering problem for levels $1/2$ and $3/2$ of a given concatenation hierarchy, compute precisely the generalized pointlike sets for the corresponding pseudovarieties of ordered monoids $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$. The ω-reducibility of the pseudovarieties $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$ was proven in [37] using these algorithms from [25].

In this paper, we prove the same result – the ω-reducibility of the pseudovarieties $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$, but now purely algebraically. No knowledge of the covering of regular languages is needed. Even the knowledge of the generalized $\mathsf{V}_{1/2}$-pointlike and $\mathsf{V}_{3/2}$-pointlike sets is not needed for this proof. We use some ideas from the proofs of [25], primarily

- a stratification of a polynomial closure of a lattice of regular languages into a sequence of finite lattices of regular languages $\rightarrow$ a stratification of a polynomial closure of a pseudovariety of ordered monoids into a sequence of locally finite pseudovarieties of ordered monoids,
- factorization forests invented by Simon [35].

However, our proof is independent of the proofs of [25] and it is also still considerably different from those proofs – both due to different approaches (language theoretic, in terms of $\mathcal{C}$-covering, versus algebraic, in terms of pseudoinequalities valid in a pseudovariety) and different aims (to prove the correctness of the algorithm vs. to prove the ω-reducibility of pseudovarieties).

Furthermore, the algorithms from [25] computing the generalized $\mathsf{V}_{1/2}$-pointlike and $\mathsf{V}_{3/2}$-pointlike sets can be recovered from our proofs in this paper. More precisely, in our proofs, we use results from [37] that correspond to the proofs of the soundness of the algorithms from [25]. From the proofs of this paper, the proofs of the completeness of those algorithms can be recovered.

Another advantage of our new proofs is that we provide general lemmas for proving the ω-reducibility of potentially all half levels of a concatenation hierarchy with a locally finite pseudovariety V_0. More precisely, we prove two lemmas, let's say A and B, where Lemma B can be seen as an extension of Lemma A to a more complicated structure. Then,

- for the proof of the ω-reducibility of the pseudovariety $\mathsf{V}_{1/2}$, we use Lemma A;
- for the proof of the ω-reducibility of the pseudovariety $\mathsf{V}_{3/2}$, we use Lemmas A and B;
- for potential proofs of the ω-reducibility of higher half levels, we would use Lemmas A and B and some other lemmas, which would be extensions of Lemmas A and B…

To sum up, we prove the ω-reducibility of the pseudovarieties $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$

- using only algebraic techniques of the theory of finite monoids,
- not using the knowledge of the generalized $\mathsf{V}_{1/2}$-pointlike and $\mathsf{V}_{3/2}$-pointlike sets, respectively, but these generalized pointlike sets can be obtained from our proofs,
- using general theorems, which could be potentially used also for proofs of the ω-reducibility of higher half levels.

The content of this paper is based on the author's PhD thesis [36]. However, we use a new concept of the presentation in this paper. We use a more general structure of the main proofs for the process to be more understandable for the reader and more convenient for its potential use for higher half levels of concatenation hierarchies. To get this, the following modifications of the material from the PhD thesis [36] have been made:

- all needed propositions concerning the pseudovarieties $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$ have been generalized if possible to cover both cases,
- main proofs have been split into general lemmas and specific theorems (for individual pseudovarieties $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$) using these general lemmas,
- for the proof of the ω-reducibility of the pseudovariety $\mathsf{V}_{3/2}$, a new notion of *k-factoriality* is used to make the proof a bit simpler and more intelligible.

2 Preliminaries

In this section, we recall briefly notions and statements regarding pseudovarieties of ordered monoids, pointlike sets and ω-reducibility, concatenation hierarchies, factorization forests, and related topics that are used later in the paper. This section partially coincides with Preliminaries of the author's paper [37], where more details can be found. For deeper familiarization with the subject, the author suggests, e.g., the survey paper [21], the book chapter [19], and the books [1, 32].

2.1 Pseudovarieties of ordered monoids and ways of their description

2.1.1 Basic definitions

Let M be a set. A *preorder* $\lesssim$ on M is a reflexive and transitive binary relation on M. A preorder $\lesssim$ on a monoid M is called *compatible* if, for all $s_1, s_2, t_1, t_2 \in M$, the following implication holds: $(s_1 \lesssim t_1, s_2 \lesssim t_2) \Rightarrow s_1s_2 \lesssim t_1t_2$. An *ordered monoid* is a monoid equipped with a compatible partial order.

Let $\lesssim$ be a compatible preorder on a monoid M. Then the corresponding equivalence relation $\sim$ associated with $\lesssim$ is a congruence on M, and the quotient monoid $M/\sim$, naturally equipped with the partial order $\leq$ induced by preorder $\lesssim$, is an ordered monoid. We denote by $[s]_\sim$ the corresponding equivalence class of an element $s \in M$.

A *pseudovariety of ordered monoids* is a nonempty class of finite ordered monoids closed under taking ordered submonoids, images in homomorphisms into finite ordered monoids, and finite direct products of ordered monoids.

Let V be a pseudovariety of ordered monoids. Then the *dual pseudovariety* of V, denoted by V^d, consists of monoids from V equipped with the dual order, i.e.,

$$(M, \leq) \in \mathsf{V} \Leftrightarrow (M, \geq) \in \mathsf{V}^\mathrm{d}.$$

A pseudovariety of ordered monoids V is *selfdual* if the property $\mathsf{V}^\mathrm{d} = \mathsf{V}$ holds.

2.1.2 Pseudowords and pseudoinequalities

Let A be a (finite) alphabet. Due to our purposes, all alphabets considered in this paper are supposed to be finite. We denote by $\overline{\Omega}_A\mathsf{M}$ the free profinite monoid generated by A. It is the completion of the free monoid A^* equipped with a specific metric (see details, e.g., in [21]). The metric monoid $\overline{\Omega}_A\mathsf{M}$ is compact. The elements of $\overline{\Omega}_A\mathsf{M}$ are called *pseudowords*. Finite monoids are considered to be equipped with the discrete metric. Then every homomorphism $\alpha\colon A^* \to M$ to a finite monoid M can be uniquely extended to a continuous homomorphism $\widehat{\alpha}\colon \overline{\Omega}_A\mathsf{M} \to M$.

For every pseudoword $u \in \overline{\Omega}_A\mathsf{M}$, the sequence $\{u^{n!}\}_{n=1}^{\infty}$ is convergent. Its limit, denoted by u^ω, is an idempotent of $\overline{\Omega}_A\mathsf{M}$. One can view $\_^\omega\colon u \mapsto u^\omega$ as a unary operation on $\overline{\Omega}_A\mathsf{M}$. Denote by ω the signature $\omega = \{\_ \cdot \_, 1, \_^\omega\}$ constituted by the binary symbol $\_ \cdot \_$, the nullary symbol 1, and the unary symbol $\_^\omega$. Then the monoid $\overline{\Omega}_A\mathsf{M}$ can be viewed as an ω-algebra, where the symbols $\_ \cdot \_$, 1, and $\_^\omega$ are interpreted as the operation of multiplication on $\overline{\Omega}_A\mathsf{M}$, the neutral element of $\overline{\Omega}_A\mathsf{M}$, and the operation of ω-power defined above, respectively. We denote $\Omega_A^\omega\mathsf{M}$ the ω-subalgebra of the ω-algebra $\overline{\Omega}_A\mathsf{M}$ generated by the alphabet A. Elements of $\Omega_A^\omega\mathsf{M}$ are called *ω-words*.

A *pseudoinequality* over an alphabet A is a pair of pseudowords $(u, v) \in \overline{\Omega}_A\mathsf{M} \times \overline{\Omega}_A\mathsf{M}$, denoted by $u \leq v$. Similarly, a *pseudoidentity* over A is a pair of pseudowords $(u, v) \in \overline{\Omega}_A\mathsf{M} \times \overline{\Omega}_A\mathsf{M}$, denoted by $u = v$. A pseudoidentity $u = v$ can be seen as a pair of pseudoinequalities $u \leq v$ and $v \leq u$. Let V be a pseudovariety of ordered monoids. A pseudoinequality $u \leq v$ is *valid* or *satisfied* in V, and we write $\mathsf{V} \models u \leq v$, if for every homomorphism $\alpha\colon A^* \to M$, the inequality $\widehat{\alpha}(u) \leq \widehat{\alpha}(v)$ holds.

Let I be a set of pseudoinequalities. We denote by $[\![I]\!]$ the class of all finite ordered monoids that satisfy all the pseudoinequalities from I. It is known that a nonempty class of finite ordered monoids V is a pseudovariety if and only if there exists a set of pseudoinequalities defining V [17, 22].

Denote by $\lesssim_\mathsf{V}^A$ the set of all pseudoinequalities over A that are valid in a pseudovariety V. The relation $\lesssim_\mathsf{V}^A$ is a compatible preorder on $\overline{\Omega}_A\mathsf{M}$. Denote by $\sim_\mathsf{V}^A$ the congruence on $\overline{\Omega}_A\mathsf{M}$ associated with the compatible preorder $\lesssim_{\mathsf{V}}^A$. It is the set of all pseudoidentities over A that are valid V. The quotient monoid $\overline{\Omega}_A\mathsf{M}/\sim_{\mathsf{V}}^A$ is the free pro-V monoid over A and is denoted by $\overline{\Omega}_A\mathsf{V}$. It can be viewed also as the completion of the monoid $A^*/\sim_{\mathsf{V}}^A|_{A^*\times A^*}$ equipped with a specific metric (see details, e.g., in [21]). The free profinite monoid $\overline{\Omega}_A\mathsf{M}$ is a special case of $\overline{\Omega}_A\mathsf{V}$, with $\mathsf{V}=\mathsf{M}$ being the pseudovariety of all finite monoids. We denote by π_{V}^A the natural projection $\pi_{\mathsf{V}}^A\colon \overline{\Omega}_A\mathsf{M}\to\overline{\Omega}_A\mathsf{V}=\overline{\Omega}_A\mathsf{M}/\sim_{\mathsf{V}}^A$.

A pseudovariety of ordered monoids V is called *locally finite* if, for every alphabet A, the monoid $\overline{\Omega}_A\mathsf{V}$ is finite.

2.1.3 Positive varieties of regular languages

Let A be an alphabet. A *lattice of regular languages* over A is a set $\mathcal{C}\subseteq 2^{A^*}$ of regular languages that contains languages $\emptyset$, A^* and is closed under union and intersection. A lattice of regular languages is called *quotienting* if it is closed under quotients. A *class of regular languages* is an assignment $\mathcal{V}$ that assigns to every alphabet A a set of regular languages $\mathcal{V}(A)\subseteq 2^{A^*}$. A *positive variety* of regular languages is a class of regular languages $\mathcal{V}$ such that every set $\mathcal{V}(A)$ is a quotienting lattice and that satisfies the condition that, for every homomorphism $\varphi\colon A^*\to B^*$, if a language L belongs to $\mathcal{V}(B)$, then the language $\varphi^{-1}(L)$ belongs to $\mathcal{V}(A)$. A *variety* of regular languages is a positive variety $\mathcal{V}$ satisfying that all quotienting lattices $\mathcal{V}(A)$ are closed also under complementation.

There is a one-to-one correspondence between varieties of regular languages and selfdual pseudovarieties of ordered monoids [15] and, more generally, a one-to-one correspondence between positive varieties of regular languages and pseudovarieties of ordered monoids [18]. In this correspondence, a positive variety of regular languages $\mathcal{V}$ corresponds to the pseudovariety generated by syntactic ordered monoids of languages from $\mathcal{V}(A)$ for an arbitrary A.

A class of regular languages $\mathcal{V}$ is called *locally finite* if, for every alphabet A, the set $\mathcal{V}(A)$ is finite. A positive variety of regular languages is locally finite if and only if the corresponding pseudovariety of ordered monoids is locally finite.

For every language $L\subseteq A^*$, denote by $\overline{L}$ the topological closure of L in $\overline{\Omega}_A\mathsf{M}$. For every quotienting lattice of regular languages $\mathcal{C}\subseteq 2^{A^*}$, we define a binary relation $\lesssim_{\mathcal{C}}\subseteq\overline{\Omega}_A\mathsf{M}\times\overline{\Omega}_A\mathsf{M}$ in the following way:

$$u\lesssim_{\mathcal{C}} v\quad\Leftrightarrow\quad\big(\forall L\in\mathcal{C}\colon u\in\overline{L}\ \Rightarrow\ v\in\overline{L}\big).$$

The relation $\lesssim_{\mathcal{C}}$ is a compatible preorder on $\overline{\Omega}_A\mathsf{M}$.

The following proposition shows a connection between the defined relations $\lesssim_{\mathsf{V}}^A$ and $\lesssim_{\mathcal{V}(A)}$, where $\mathcal{V}$ is the positive variety of regular languages corresponding to a pseudovariety of ordered monoids V.

Proposition 2.1 ([5, Proposition 2.4]). *Let $u,v\in\overline{\Omega}_A\mathsf{M}$ be pseudowords, V be a pseudovariety of ordered monoids, and $\mathcal{V}$ be the corresponding positive variety of regular languages. The pseudoinequality $u\le v$ is valid in V if and only if the property $u\lesssim_{\mathcal{V}(A)} v$ holds.*

By the preceding proposition, we obtain the equality $\lesssim_{\mathsf{V}}^{A} = \lesssim_{\mathcal{V}(A)}$.

2.2 Pointlike sets and ϖ-reducibility

2.2.1 Pointlike sets

The original definition of V-*pointlike sets*, where V is a pseudovariety of *ordinary* monoids, can be found, e.g., in [32, p. 82][1]. We use an equivalent version of the original definition. It has appeared in [23, Theorem 3.3], where the equivalence between the two versions has been proven.

Let V be a pseudovariety of monoids, M be a finite monoid, and $\alpha \colon A^* \to M$ be an arbitrary surjective homomorphism, where A is an arbitrary alphabet. Let $S = \{s_1, \dots, s_n\}$ be a subset of M. We say that the set $S \subseteq M$ is V-*pointlike* if there exist pseudowords $u_1 \in \widehat{\alpha}^{-1}(s_1), \dots, u_n \in \widehat{\alpha}^{-1}(s_n)$ such that the pseudoidentities $u_1 = u_2 = \cdots = u_n$ are valid in V. The definition is independent of the choice of a homomorphism α onto M (see, e.g., [23, Theorem 3.3]).

In this paper, we are concerned with two-element V-pointlike sets $\{s, t\}$ and with its adjustment for our framework of pseudovarieties of *ordered* monoids. The corresponding notion (in terms of the corresponding quotienting lattices of regular languages) appeared in [26] for a special case of half levels of the Straubing–Thérien hierarchy and later in [27] for a general case. The definition below was used in [2].

Let V be a pseudovariety of *ordered* monoids, M be a finite monoid, and $\alpha \colon A^* \to M$ be an arbitrary surjective homomorphism, where A is an arbitrary alphabet. We say that a pair of elements $(s, t) \in M \times M$ is V-*inequality-like* if there exist pseudowords $u \in \widehat{\alpha}^{-1}(s)$, $v \in \widehat{\alpha}^{-1}(t)$ such that the *pseudoinequality* $u \leq v$ is valid in V. Similarly to the case of pointlike sets, this definition is independent of the choice of a homomorphism α onto M [2, Lemma 5]. Note that V-inequality-like pairs can be seen as a generalization of two-element V-pointlike sets to all pseudovarieties V of ordered monoids since, for a selfdual pseudovariety V, a pair $(s, t) \in M \times M$ is V-inequality-like if and only if the corresponding two-element set $\{s, t\} \subseteq M$ is V-pointlike.

We denote by $P_{\mathsf{V}}[M]$ the set of all V-inequality-like pairs of a finite monoid M. Note that $P_{\mathsf{V}}[M]$ is a submonoid of the monoid $M \times M$.

We will need a further generalization of the already established concepts. In what follows, we will consider sets of the form 2^M, where M is a monoid, to be equipped with the "pointwise" multiplication:

$$\forall S, T \in 2^M \colon \quad S \cdot T = \{s \cdot t \mid s \in S, t \in T\}.$$

Let $\mathcal{P}_{\mathsf{V}}[M]$ be the set of all pairs $(s, S) \in M \times 2^M$, where $S = \{s_1, \dots, s_n\}$, such that there exist pseudowords $u \in \widehat{\alpha}^{-1}(s)$ and $v_i \in \widehat{\alpha}^{-1}(s_i)$, $i = 1, \dots, n$, satisfying $\mathsf{V} \models u \leq v_i$ for every i.

[1] More precisely, there is a definition of V-pointlike sets for V being a pseudovariety of *semigroups* in [32]. The definition for V being a pseudovariety of *monoids* is analogous.

Note that $\mathcal{P}_{\mathsf{V}}[M]$ is a submonoid of the monoid $M \times 2^M$ and also a submonoid of the monoid of all pairs $(s, S) \in M \times 2^M$ satisfying that, for every $t \in S$, (s, t) is a V-inequality-like pair.

Further, the sets $\mathcal{P}_{\mathsf{V}}[M]$ can be considered as a generalization of V-pointlike sets to all pseudovarieties V of ordered monoids. Indeed, if V is a selfdual pseudovariety of ordered monoids, then, by the definition of $\mathcal{P}_{\mathsf{V}}[M]$, $\big(s, \{s_1, \ldots, s_n\}\big) \in \mathcal{P}_{\mathsf{V}}[M]$ if and only if $\{s, s_1, \ldots, s_n\}$ is a V-pointlike set of M.

2.2.2 ω-reducibility

A notion of *ω-reducibility* of a pseudovariety of semigroups was established in [6]. Later, in [4], the same notion was used for a simplified case and it was extended also to the case of pseudovarieties of *ordered* monoids. In this paper, we use the latter definition and its extension to sets of the form $P_{\mathsf{V}}[M]$ and $\mathcal{P}_{\mathsf{V}}[M]$.

Let V be a pseudovariety of *ordered* monoids, M be a finite monoid, and $\alpha\colon A^* \to M$ be an arbitrary surjective homomorphism, where A is an arbitrary alphabet. We say that the set $P_{\mathsf{V}}[M]$ of V-inequality-like pairs is *ω-reducible* if, for every pair $(s, t) \in P_{\mathsf{V}}[M]$, the relevant pseudowords $u \in \widehat{\alpha}^{-1}(s)$, $v \in \widehat{\alpha}^{-1}(t)$ satisfying $\mathsf{V} \models u \leq v$ can be chosen from the monoid $\Omega^{\omega}_{A}\mathsf{M}$. It can be proven similarly to [2, Lemma 5] that this definition is also independent of the choice of a homomorphism α onto M.

Further, we say that the pseudovariety V is *ω-reducible* if, for every finite monoid M, the set $P_{\mathsf{V}}[M]$ is ω-reducible.

Finally, we extend the notion of *ω-reducibility* to the sets $\mathcal{P}_{\mathsf{V}}[M]$. We say that the set $\mathcal{P}_{\mathsf{V}}[M]$ is *ω-reducible* if, for every pair $(s, S) \in \mathcal{P}_{\mathsf{V}}[M]$, there exists an ω-word $u \in \widehat{\alpha}^{-1}(s)$ such that, for every $t \in S$, there exists an ω-word $v \in \widehat{\alpha}^{-1}(t)$ having the property that $\mathsf{V} \models u \leq v$.

2.3 Concatenation hierarchies

2.3.1 Definition and properties

Let $\mathcal{V}$ be a class of regular languages. The *polynomial closure* of the class $\mathcal{V}$ is a class of regular languages $Pol(\mathcal{V})$ such that, for every alphabet A, the set $Pol(\mathcal{V})(A)$ consists precisely of all finite unions of languages of the form $L_0a_1L_1 \cdots a_nL_n$, where $a_i \in A$, $L_i \in \mathcal{V}(A)$. The *Boolean closure* of the class $\mathcal{V}$ is a class of regular languages $Bool(\mathcal{V})$ such that, for every alphabet A, the set $Bool(\mathcal{V})(A)$ is the closure of $\mathcal{V}(A)$ under the Boolean operations.

A *concatenation hierarchy* of regular languages consists of integer and half levels, built of classes of regular languages. Level 0, called the *basis* of the hierarchy, is an arbitrary variety of regular languages $\mathcal{V}_0$. Then, for every nonnegative integer n, $\mathcal{V}_{n+1/2}$ is the polynomial closure of $\mathcal{V}_n$, and $\mathcal{V}_{n+1}$ is the Boolean closure of $\mathcal{V}_{n+1/2}$.

Let $\mathcal{V}$ be a class of regular languages. We denote by Co-$\mathcal{V}$ a class of regular languages such that, for every alphabet A, the set Co-$\mathcal{V}(A)$ consists precisely of complements of languages from $\mathcal{V}(A)$. If $\mathcal{V}$ is a positive variety of regular languages corresponding to a pseudovariety of ordered monoids V, then Co-$\mathcal{V}$ is a positive variety corresponding to the dual pseudovariety V^{d}. For $n \geq 1$, the class $\mathcal{V}_{n+1/2}$ can be equivalently defined as the polynomial closure of the class Co-$\mathcal{V}_{n-1/2}$ [27][2].

It is known that the polynomial closure of a positive variety of regular languages is also a positive variety of regular languages [20, Theorem 5.4]. It is an easy exercise to show that the Boolean closure of a positive variety is a variety of regular languages. These properties imply that all half levels of a concatenation hierarchy are positive varieties and all integer levels are varieties of regular languages.

Further, it is known that every concatenation hierarchy with a locally finite basis $\mathcal{V}_0$ is infinite, more precisely that, for every integer level n and for every alphabet A satisfying $|A| > 1$, the strict inclusions $\mathcal{V}_n(A) \subsetneq \mathcal{V}_{n+1/2}(A) \subsetneq \mathcal{V}_{n+1}(A)$ hold [27][3].

2.3.2 The corresponding hierarchy of pseudovarieties

Due to the correspondence between positive varieties of regular languages and pseudovarieties of ordered monoids, there is a parallel hierarchy of pseudovarieties of ordered monoids $\mathsf{V}_0 \subseteq \mathsf{V}_{1/2} \subseteq \mathsf{V}_1 \subseteq \cdots$, where V_0 is a selfdual pseudovariety, called the *basis* of the hierarchy.

Let V be a pseudovariety of ordered monoids corresponding to a positive variety of regular languages $\mathcal{V}$. We denote by $\mathrm{Pol}\,\mathsf{V}$ the pseudovariety of ordered monoids corresponding to the positive variety $Pol(\mathcal{V})$.

The following characterization of $\mathrm{Pol}\,\mathsf{V}$ by pseudoinequalities appeared in a slightly different version in [11][4] In [2], an equivalence of the version from [11] and the version stated here was shown without linking them explicitly to the polynomial closure. A link between the version stated here and the polynomial closure appeared in [27].

Proposition 2.2

$$\mathrm{Pol}\,\mathsf{V} = [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid u, v \in \overline{\Omega}_A \mathsf{M} \text{ for some } A,\ \mathsf{V} \models u \leq v]\!].$$

Let $n \in \mathbb{N}$. By the alternative description of half levels of a concatenation hierarchy $\mathcal{V}_{n+1/2} = Pol(\text{Co-}\mathcal{V}_{n-1/2})$, we obtain the following relation for the corresponding pseudovarieties:

$$\mathsf{V}_{n+1/2} = \mathrm{Pol}(\mathsf{V}_{n-1/2})^{\mathrm{d}}.$$

2.3.3 Straubing–Thérien hierarchy

The *Straubing–Thérien hierarchy* is a concatenation hierarchy with the basis $\mathcal{V}_0$ being the trivial variety of regular languages. It is known that the level $3/2$ of the Straubing–Thérien hierarchy is the level $1/2$ of another concatenation hierarchy with the basis $\mathcal{W}_0$

[2] The paper [27] deals with generalized concatenation hierarchies, where levels are *quotienting lattices* of regular languages over a *fixed alphabet*. The property of a non-generalized concatenation hierarchy follows from the corresponding properties of the generalized hierarchies over individual alphabets.

[3] The papers [11] and [27] deal with *quotienting lattices* of regular languages $\mathcal{C}$ over a *fixed alphabet*. The description of $\mathrm{Pol}\,\mathsf{V}$ by pseudoinequalities follows from the corresponding description of the polynomial closure of $\mathcal{C}$ from [11] and [27], respectively.

[4] The papers [11] and [27] deal with *quotienting lattices* of regular languages $\mathcal{C}$ over a *fixed alphabet*. The description of $\mathrm{Pol}\,\mathsf{V}$ by pseudoinequalities follows from the corresponding description of the polynomial closure of $\mathcal{C}$ from [11] and [27], respectively.

being the variety of regular languages corresponding to the pseudovariety Sl of finite semilattices [7].

Since the basis of the Straubing–Thérien hierarchy consists only of star-free languages, all levels consist of star-free languages as well, by the definition of concatenation hierarchies. The variety of star-free languages corresponds to the pseudovariety A of aperiodic monoids, which has a characterization $\mathsf{A} = [\![a^{\omega+1} = a^{\omega}]\!]$ [33].

Further, it is known that the free pro-Sl monoid $\overline{\Omega}_A\mathsf{Sl}$ is isomorphic to $(2^A, \cup)$. For every word $u \in A^*$, denote the content of u by $c(u)$. Then the *content function* $c\colon A^* \to (2^A, \cup)$ is a homomorphism and its continuous extension $\widehat{c}\colon \overline{\Omega}_A\mathsf{M} \to (2^A, \cup)$ equals the natural projection π_{Sl}.

Finally, we mention known descriptions of low levels of the Straubing–Thérien hierarchy by *ω-inequalities* or *ω-identities*. An *ω-inequality* (*ω-identity*) is a pseudoinequality $u \leq v$ (pseudoidentity $u = v$), where both u, v are ω-words.

We know the following characterizations of levels 1/2, 1, 3/2, and 2, 5/2, 7/2 of the Straubing–Thérien hierarchy by ω-inequalities or ω-identities ([7, 24, 34], and [37], respectively):

- $\mathsf{V}_{1/2} = [\![1 \leq a]\!]$,
- $\mathsf{V}_1 = [\![(ab)^{\omega}a = (ab)^{\omega}, b(ab)^{\omega} = (ab)^{\omega}]\!]$,
- $\mathsf{V}_{3/2} = [\![u^{\omega} \leq u^{\omega}vu^{\omega} \mid u, v \in A^* \text{ for some } A,\ c(u) = c(v)]\!]$,
-
$$\begin{aligned}\mathsf{V}_2 = [\![&(v_1 y\, \overline{v}_1 \overline{y})^{\omega} u_1 x^{\omega} u_2 (\overline{z}\, \overline{v}_2 z\, v_2)^{\omega} = (v_1 y\, \overline{v}_1 \overline{y})^{\omega} v_1 y\, \overline{u}_1 (\overline{x})^{\omega} \overline{u}_2 z\, v_2 (\overline{z}\, \overline{v}_2 z\, v_2)^{\omega} \mid \\ & x, \overline{x}, y, \overline{y}, z, \overline{z}, u_1, \overline{u}_1, u_2, \overline{u}_2, v_1, \overline{v}_1, v_2, \overline{v}_2 \in \Omega_A^{\omega}\mathsf{M} \text{ for some } A,\\ & \widehat{c}(x) = \widehat{c}(u_1 x\, u_2) = \widehat{c}(\overline{x}) = \widehat{c}(\overline{u}_1 \overline{x}\, \overline{u}_2),\\ & \mathsf{V}_{3/2} \models x^{\omega} \leq y, \mathsf{V}_{3/2} \models x^{\omega} \leq z, \mathsf{V}_{3/2} \models (\overline{x})^{\omega} \leq \overline{y}, \mathsf{V}_{3/2} \models (\overline{x})^{\omega} \leq \overline{z},\\ & \mathsf{V}_{3/2} \models u_i \leq v_i, \mathsf{V}_{3/2} \models \overline{u}_i \leq \overline{v}_i \text{ for } i = 1, 2]\!],\end{aligned}$$
- $\mathsf{V}_{5/2} = [\![u^{\omega} \leq u^{\omega}vu^{\omega} \mid u, v \in \Omega_A^{\omega}\mathsf{M} \text{ for some } A,\ \mathsf{V}_{3/2} \models v \leq u]\!]$,
- $\mathsf{V}_{7/2} = [\![u^{\omega} \leq u^{\omega}vu^{\omega} \mid u, v \in \Omega_A^{\omega}\mathsf{M} \text{ for some } A,\ \mathsf{V}_{5/2} \models v \leq u]\!]$.

Remark 2.3 By [30, Corollary 36 and Remark 37] and [37, Corollary 4.4 and Theorem 5.1], we obtain an alternative basis of ω-identities for the pseudovariety V_2:

$$\begin{aligned}\mathsf{V}_2 = [\![&(x^{\omega}py^{\omega}qx^{\omega})^{\omega}(x^{\omega}ry^{\omega}sx^{\omega})^{\omega} = (x^{\omega}py^{\omega}qx^{\omega})^{\omega}x^{\omega}py^{\omega}sx^{\omega}(x^{\omega}ry^{\omega}sx^{\omega})^{\omega},\\ & u^{\omega} = u^{\omega}vu^{\omega} \mid \\ & x, y, p, q, r, s \in A^*,\ u, v \in \Omega_A^{\omega}\mathsf{M} \text{ for some } A,\\ & c(x) = c(y) = c(p) = c(q) = c(r) = c(s),\ \mathsf{V}_{3/2} \models v \leq u]\!].\end{aligned}$$

2.4 Factorization forests

This subsection presents a standard tool in the theory of finite semigroups, invented by Simon [35]. The presentation is borrowed from the author's PhD thesis [36].

For every monoid M, we denote by $E(M)$ the set of all idempotents of M. Let A be a finite alphabet, M be a finite monoid, $\alpha\colon A^* \to M$ be a homomorphism. By ε, we denote the empty word, as usual. A *factorization forest* for α is a function d that assigns to every word

$$u \in A^* \setminus (A \cup \{\varepsilon\})$$

a finite sequence of nonempty words $d(u) = (u_1, u_2, \ldots, u_n)$ such that the following conditions are satisfied:

1. $u = u_1 \cdots u_n$,
2. if $n \geq 3$, then there exists an idempotent $e \in E(M)$ such that we have $\alpha(u_i) = e$ for every $i \in \{1, \ldots, n\}$.

For every word $u \in A^*$ we define its *height* $h_d(u)$ in the factorization forest d inductively:

- If $u \in A \cup \{\varepsilon\}$, then $h_d(u) = 0$.
- If $d(u) = (u_1, \ldots, u_n)$, then $h_d(u) = \max\{h_d(u_i) \mid 1 \leq i \leq n\} + 1$.

Further, the *height* of a factorization forest d is defined in the following way:

$$h(d) := \sup\{h_d(u) \mid u \in A^*\}.$$

Theorem 2.4 ([16, 35]). *Let A be an alphabet, M be a finite monoid, $\alpha\colon A^* \to M$ be a homomorphism. Then there exists a factorization forest d for α of height at most* $3|M| - 1$.

3 Locally finite pseudovarieties

In this section, we show how we can use locally finite pseudovarieties for proofs of the ω-reducibility. The main aim of this paper is to prove the ω-reducibility of pseudovarieties $\mathsf{V}_{1/2}$ and $\mathsf{V}_{3/2}$ corresponding to levels $1/2$ and $3/2$, respectively, of a concatenation hierarchy with a locally finite basis. Recall that a pseudovariety V is ω-reducible if, for every finite monoid M, the set of inequality-like pairs $P_{\mathsf{V}}[M] \subseteq M \times M$ is ω-reducible. However, we are not able to prove the ω-reducibility of $P_{\mathsf{V}_{1/2}}[M]$ and $P_{\mathsf{V}_{3/2}}[M]$ directly. To obtain the ω-reducibility of these sets, we actually need to prove the ω-reducibility of the more general sets $\mathcal{P}_{\mathsf{V}_{1/2}}[M] \subseteq M \times 2^M$ and $\mathcal{P}_{\mathsf{V}_{3/2}}[M] \subseteq M \times 2^M$. Hence, in the whole paper, not excluding this section, we will concentrate primarily on properties of sets of the form $\mathcal{P}_{\mathsf{V}}[M]$.

In the first subsection, we define a W-*compatible* homomorphism for a locally finite pseudovariety W and we recall statements from the author's paper [37] needed for our purposes. Specifically, we know from [37] that, when proving the ω-reducibility of a set of the form $\mathcal{P}_{\mathsf{V}}[M]$, we can restrict, without loss of generality, to finite monoids M for which a surjective W-*compatible* homomorphism of the form $A^* \to M$ exists.

In the second subsection, we state a crucial theorem showing the way how to prove the ω-reducibility of a set of the form $\mathcal{P}_{\mathsf{V}}[M]$ using a locally finite pseudovariety $\mathsf{W} \subseteq \mathsf{V}$.

In the third subsection, we introduce *stratifications* of pseudovarieties. We prove their basic properties with respect to sets of the form $\mathcal{P}_{\mathsf{V}}[M]$. Then we introduce specific stratifications for pseudovarieties of the forms $\operatorname{Pol}\mathsf{V}$ and $\operatorname{Pol}(\operatorname{Pol}\mathsf{V})^{\mathrm{d}}$. If we consider a concatenation hierarchy with $\mathsf{V}_0 = \mathsf{V}$, then the pseudovarieties $\mathrm{Pol}\,\mathsf{V}$ and $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ correspond to levels $1/2$ and $3/2$, respectively, of this concatenation hierarchy.

3.1 Restriction to a W-compatible homomorphism

The content of this subsection comes from the author's paper [37]. We recall it here since it is necessary for our proofs of the ω-reducibility.

3.1.1 Basic definitions and a connection with ω-reducibility

Let W be a pseudovariety of ordered monoids and $\alpha\colon A^* \to M$ be a homomorphism into a finite monoid M. We say that the homomorphism α is W-*compatible* if the inclusion $\ker(\widehat{\alpha}) \subseteq \ker(\pi_{\mathsf{W}}^{A})$ holds, where ker stands for the kernel of a homomorphism, as usual. Note that, if there exists a W-compatible homomorphism $\alpha_A\colon A^* \to M$ for every alphabet A, then the monoid $\overline{\Omega}_A\mathsf{W}$ is finite for every A, i.e., the pseudovariety W is locally finite.

Let W be a locally finite pseudovariety of ordered monoids. A W-*completion*[5] of a homomorphism $\alpha\colon A^* \to M$ is the homomorphism $\alpha_{\mathsf{W}}\colon A^* \to M \times \overline{\Omega}_A\mathsf{W}$ defined by setting $\alpha_{\mathsf{W}}(u) = \big(\alpha(u), \pi_{\mathsf{W}}^{A}(u)\big)$ for every word $u \in A^*$. Then $\widehat{\alpha_{\mathsf{W}}}(u) = \big(\widehat{\alpha}(u), \pi_{\mathsf{W}}^{A}(u)\big)$ for every pseudoword $u \in \overline{\Omega}_A\mathsf{M}$. The W-completion is clearly W-compatible. Denote by $M_{\alpha_{\mathsf{W}}}$ the homomorphic image of α_{W}.

Let V be an arbitrary pseudovariety of ordered monoids. The following lemma shows that, when proving the ω-reducibility of $\mathcal{P}_{\mathsf{V}}[M]$ for every finite monoid M, we can restrict, without loss of generality, to finite monoids of the form $M_{\alpha_{\mathsf{W}}}$.

Lemma 3.1 ([37, Lemma 4.1]). *Let* W *be a locally finite pseudovariety,* V *be an arbitrary pseudovariety of ordered monoids,* M *be a finite monoid, and* $\alpha\colon A^* \to M$ *be a surjective homomorphism. If the set* $\mathcal{P}_{\mathsf{V}}[M_{\alpha_{\mathsf{W}}}]$ *is* ω*-reducible, then the set* $\mathcal{P}_{\mathsf{V}}[M]$ *is also* ω*-reducible.*

3.1.2 Word reducibility

Let V be a pseudovariety of ordered monoids and $\alpha\colon A^* \to M$ be a surjective homomorphism into a finite monoid M. We say that the set $\mathcal{P}_{\mathsf{V}}[M]$ is *word reducible* if, for every pair $(s, S) \in \mathcal{P}_{\mathsf{V}}[M]$, there exists a *word* $u \in \alpha^{-1}(s)$ such that, for every $t \in S$, there exists a *word* $v \in \alpha^{-1}(t)$ having the property that $\mathsf{V} \models u \leq v$.

The following lemma is analogous to Lemma 3.1.

Lemma 3.2 ([37, Lemma 4.2]). *Let* W *be a locally finite pseudovariety,* V *be an arbitrary pseudovariety of ordered monoids, and* $\alpha\colon A^* \to M$ *be a surjective homomorphism onto a finite monoid* M*. If the set* $\mathcal{P}_{\mathsf{V}}[M_{\alpha_{\mathsf{W}}}]$ *is word reducible, then the set* $\mathcal{P}_{\mathsf{V}}[M]$ *is also word reducible.*

[5] This notion is a translation of the corresponding notion from [25] into our context.

An application of Lemma 3.2 follows. Its version concerning only pseudovarieties of *ordinary* monoids can be seen as a special case of [6, Theorem 4.18].

Corollary 3.3 ([37, Corollary 4.4]). *Let* W *be a locally finite pseudovariety of ordered monoids. For every finite monoid* M*, the set* $\mathcal{P}_{\mathsf{W}}[M]$ *is word reducible.*

3.2 A way for proving ω-reducibility

Let V be a pseudovariety of ordered monoids. Let $u \in \overline{\Omega}_A\mathsf{M}$ be a pseudoword and $U \subseteq \overline{\Omega}_A\mathsf{M}$ be a set of pseudowords. In what follows, we will use the simplified notation

$$\mathsf{V} \models u \leq U$$

for a system of inequalities

$$\{\mathsf{V} \models u \leq w \mid w \in U\}.$$

and similarly the notation $\mathsf{V} \models u = U$ instead of a system of equalities

$$\{\mathsf{V} \models u = w \mid w \in U\}.$$

The following proposition gives a guide for proofs of the ω-reducibility of sets of the form $\mathcal{P}_{\mathsf{V}}[M]$, where the pseudovariety V is not locally finite. It comes from the author's PhD thesis [36].

Proposition 3.4 *Let* M *be a finite monoid and* $\alpha\colon A^* \to M$ *be a surjective homomorphism. Let* V *be a pseudovariety of ordered monoids and* W *be a locally finite pseudovariety such that* $\mathsf{W} \subseteq \mathsf{V}$ *and, for every word* $u \in A^*$ *and every set* $U \subseteq A^*$ *satisfying the relation* $\mathsf{W} \models u \leq U$*, there exist an* ω*-word* $x \in \Omega_A^{\omega}\mathsf{M}$ *and a set of* ω*-words* $X \subseteq \Omega_A^{\omega}\mathsf{M}$ *satisfying the following conditions:*

i) $\mathsf{V} \models x \leq X$,
ii) $\widehat{\alpha}(x) = \alpha(u)$, $\widehat{\alpha}(X) = \alpha(U)$.

Then $\mathcal{P}_{\mathsf{V}}[M] = \mathcal{P}_{\mathsf{W}}[M]$ *and the set* $\mathcal{P}_{\mathsf{V}}[M]$ *is* ω*-reducible.*

Proof Since we have $\mathsf{W} \subseteq \mathsf{V}$ by the assumption, we obtain that $\mathcal{P}_{\mathsf{V}}[M] \subseteq \mathcal{P}_{\mathsf{W}}[M]$ by the definition of $\mathcal{P}_{\mathsf{V}}[M]$. We prove the opposite inclusion. Let $(s, S) \in \mathcal{P}_{\mathsf{W}}[M]$ be an arbitrary pair. Since W is a locally finite pseudovariety, the set $\mathcal{P}_{\mathsf{W}}[M]$ is word reducible by Corollary 3.3. This means that there exist a word $u \in A^*$ and a set $U \subseteq A^*$ satisfying the following conditions:

- $\mathsf{W} \models u \leq U$,
- $\alpha(u) = s, \alpha(U) = S$.

Then, by the assumption of Proposition 3.4, there exist an ω-word $x \in \Omega_A^{\omega}\mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^{\omega}\mathsf{M}$ satisfying the following conditions:

- $\mathsf{V} \models x \leq X$,

- $\widehat{\alpha}(x) = \alpha(u) = s, \widehat{\alpha}(X) = \alpha(U) = S$.

This implies that the pair (s, S) belongs to the set $\mathcal{P}_{\mathsf{V}}[M]$. We have proved the desired inclusion $\mathcal{P}_{\mathsf{W}}[M] \subseteq \mathcal{P}_{\mathsf{V}}[M]$.

It remains to prove that the set $\mathcal{P}_{\mathsf{V}}[M]$ is ω-reducible. Let $(s, S) \in \mathcal{P}_{\mathsf{V}}[M]$, where $S = \{s_1, \ldots, s_n\}$, be an arbitrary pair. By the definition of the set $\mathcal{P}_{\mathsf{V}}[M]$, there exist pseudowords $p, q_1, \ldots, q_n \in \overline{\Omega}_A\mathsf{M}$ satisfying the following conditions:

1. $\mathsf{V} \models p \leq q_1, \ \ldots, \ \mathsf{V} \models p \leq q_n$,
2. $\widehat{\alpha}(p) = s, \widehat{\alpha}(q_1) = s_1, \ \ldots, \ \widehat{\alpha}(q_n) = s_n$.

By the definition of the ω-reducibility of $\mathcal{P}_{\mathsf{V}}[M]$, we need to prove that there exist an ω-word $x \in \Omega_A^{\omega}\mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^{\omega}\mathsf{M}$ satisfying the following conditions:

- $\mathsf{V} \models x \leq X$,
- $\widehat{\alpha}(x) = s, \widehat{\alpha}(X) = S$.

To prove this, we use the assumptions of the theorem.

Let $\beta\colon A^* \to M_\beta \subseteq M \times \overline{\Omega}_A\mathsf{W}$ be the W-completion of the homomorphism α, i.e., $\beta = \alpha_{\mathsf{W}}$. Since the homomorphisms β and $\widehat{\beta}$ have the same image, there exist words $u, v_1, \ldots, v_n \in A^*$ such that $\beta(u) = \widehat{\beta}(p)$ and $\beta(v_i) = \widehat{\beta}(q_i)$ for every $i \in \{1, \ldots, n\}$. By the definition of the homomorphism β, these equalities imply that $\alpha(u) = \widehat{\alpha}(p) = s$, $\mathsf{W} \models u = p$ and $\alpha(v_i) = \widehat{\alpha}(q_i) = s_i$, $\mathsf{W} \models v_i = q_i$ for every $i \in \{1, \ldots, n\}$. Further, since we have $\mathsf{W} \subseteq \mathsf{V}$, Condition 1 implies that the relations $\mathsf{W} \models p \leq q_1, \ldots, \mathsf{W} \models p \leq q_n$ hold. Hence we obtain

$$\mathsf{W} \models u = p \leq q_i = v_i \quad \text{for every } i \in \{1, \ldots, n\}.$$

Then, by the assumption of the proposition, there exist an ω-word $x \in \Omega_A^{\omega}\mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^{\omega}\mathsf{M}$ satisfying the following conditions:

i) $\mathsf{V} \models x \leq X$,
ii) $\widehat{\alpha}(x) = \alpha(u) = s, \widehat{\alpha}(X) = \{\alpha(v_1), \ldots, \alpha(v_n)\} = \{s_1, \ldots, s_n\} = S$.

This is precisely what we needed to prove. □

3.3 Stratification of a pseudovariety

In our proofs of the ω-reducibility, we use *stratifications* of pseudovarieties. In particular, we are interested in stratifications of pseudovarieties of the form $\mathrm{Pol}\,\mathsf{V}$ and $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, which correspond to levels $1/2$ and $3/2$, respectively, of the concatenation hierarchy with $\mathsf{V}_0 = \mathsf{V}$.

3.3.1 Definition and a connection with $\mathcal{P}_{\mathsf{V}}[M]$

Our definition of a stratification of a *pseudovariety of ordered monoids* is similar to the definition of a stratification of an infinite *quotienting lattice* from [25, Subsection 2.3 (p. 8)]. More precisely, it is a translation of the definition from [25] into terms of pseudovarieties, with an exception of not demanding the strata to be (locally) finite.

A *stratification* of a pseudovariety of ordered monoids V is a sequence of pseudovarieties $\{\mathsf{V}_k\}_{k=0}^{\infty}$ satisfying the following two conditions:

- $\mathsf{V}_k \subseteq \mathsf{V}_{k+1}$ for every $k \in \mathbb{N}_0$,
- $\bigcup_{k=0}^{\infty} \mathsf{V}_k = \mathsf{V}$.

Example 3.5 The sequence of pseudovarieties $\{\mathsf{V}_k\}_{k=0}^{\infty}$ corresponding to (integer) levels of the Straubing–Thérien hierarchy forms a stratification of the pseudovariety A of aperiodic monoids.

The following propositions are generalizations of some results of the author's master's thesis on the Straubing–Thérien hierarchy [9, Section 4.1], which explain an algebraic interpretation of results of the paper [26] by Place and Zeitoun. They are contained also in the author's PhD thesis [36, Subsection 2.4.2].

Proposition 3.6 *Let M be a finite monoid, V be a pseudovariety of ordered monoids and $\{\mathsf{V}_k\}_{k=0}^{\infty}$ be a stratification of V. Then $\mathcal{P}_{\mathsf{V}}[M] = \bigcap_{k=1}^{\infty} \mathcal{P}_{\mathsf{V}_k}[M]$.*

Proof The inclusion $\mathcal{P}_{\mathsf{V}}[M] \subseteq \bigcap_{k=0}^{\infty} \mathcal{P}_{\mathsf{V}_k}[M]$ follows from the inclusions $\mathsf{V}_k \subseteq \mathsf{V}$ for $k \in \mathbb{N}_0$ and the definition of $\mathcal{P}_{\mathsf{V}}[M]$. We prove the opposite inclusion. Let $(s, S) \in M \times 2^M$, where $S = \{t_1, \dots, t_n\}$, be a pair belonging to $\bigcap_{k=1}^{\infty} \mathcal{P}_{\mathsf{V}_k}[M]$ and $\alpha\colon A^* \to M$ be a surjective homomorphism. By the definition of $\mathcal{P}_{\mathsf{V}_k}[M]$, for every $k \in \mathbb{N}_0$, there exist pseudowords $u^k \in \widehat{\alpha}^{-1}(s)$ and $v_i^k \in \widehat{\alpha}^{-1}(t_i)$ for $i = 1, \dots, n$ such that

$$\mathsf{V}_k \models u^k \leq v_i^k \quad \text{for every } i \in \{1, \dots, n\}.^{6} \tag{2}$$

Since the metric monoid $\overline{\Omega}_A[M]$ is compact, one can choose convergent subsequences $\{u^{k_j}\}_{j=1}^{\infty}$, $\{v_1^{k_j}\}_{j=1}^{\infty}$, …, $\{v_n^{k_j}\}_{j=1}^{\infty}$ from the sequences $\{u^k\}_{k=0}^{\infty}$, $\{v_1^k\}_{k=0}^{\infty}$, …, $\{v_n^k\}_{k=0}^{\infty}$, respectively.

We need to prove that the pair (s, S) belongs to $\mathcal{P}_{\mathsf{V}}[M]$, which means that there exist pseudowords $u \in \widehat{\alpha}^{-1}(s)$ and $v_i \in \widehat{\alpha}^{-1}(t_i)$ for $i = 1, \dots, n$ such that $\mathsf{V} \models u \leq v_i$ for every i.

Let $u := \lim_{j\to\infty} u^{k_j}$, $v_1 := \lim_{j\to\infty} v_1^{k_j}$, …, $v_n := \lim_{j\to\infty} v_n^{k_j}$. Then, by the continuity of the function $\widehat{\alpha}$, we have $\widehat{\alpha}(u) = s$, $\widehat{\alpha}(v_1) = t_1$, …, $\widehat{\alpha}(v_n) = t_n$. It remains to prove that the relation $\mathsf{V} \models u \leq U$ holds, which means that, for every finite ordered monoid $N \in \mathsf{V}$ and for every homomorphism $\beta\colon A^* \to N$, the inequalities $\widehat{\beta}(u) \leq \widehat{\beta}(v_i)$ for $i = 1, \dots, n$ hold.

Let $\beta\colon A^* \to N$ be an arbitrary homomorphism to a monoid $N \in \mathsf{V}$. By the continuity of $\widehat{\beta}$, we have $\widehat{\beta}(u) = \lim_{j\to\infty} \widehat{\beta}(u^{k_j})$, $\widehat{\beta}(v_1) = \lim_{j\to\infty} \widehat{\beta}(v_1^{k_j})$, …, $\widehat{\beta}(v_n) = \lim_{j\to\infty} \widehat{\beta}(v_n^{k_j})$. Since the monoid N is finite, there exist indices $j_0, j_1, \dots, j_n \in$ such that, for every $j \geq j_0$, we have $\widehat{\beta}(u) = \widehat{\beta}(u^{k_j})$, and for every $j \geq j_i$, we have $\widehat{\beta}(v_i) = \widehat{\beta}(v_i^{k_j})$ for $i = 1, \dots, n$. Since the monoid N belongs to the pseudovariety $\mathsf{V} = \bigcup_{k=0}^{\infty} \mathsf{V}_k$, where $\mathsf{V}_0 \subseteq \mathsf{V}_1 \subseteq \mathsf{V}_2 \subseteq \cdots$, there must exist an index $j' \in \mathbb{N}_0$ such that $N \in \mathsf{V}_{k_{j'}}$ for every $j \geq j'$. Let $m := \max\{j_0, j_1, \dots, j_n, j'\}$. Using

[6] Here, the notation u^k and v_i^k does not stand for the k-power of v_i, but it just means v_i with upper index k.

Relation (2), we obtain that $\mathsf{V}_{k_m} \models u^{k_m} \leq v_i^{k_m}$ for every $i \in \{1, \ldots, n\}$. This implies that $\widehat{\beta}(u) = \widehat{\beta}(u^{k_m}) \leq \widehat{\beta}(v_i^{k_m}) = \widehat{\beta}(v_i)$ for every $i \in \{1, \ldots, n\}$, which we needed to prove. □

Proposition 3.7 *Let M be a finite monoid,* V *be a pseudovariety of ordered monoids and* $\{\mathsf{V}_k\}_{k=0}^{\infty}$ *be a stratification of* V*. Then there exists an index $k_0 \in \mathbb{N}_0$ such that* $\mathcal{P}_{\mathsf{V}_k}[M] = \mathcal{P}_{\mathsf{V}}[M]$ *for every $k \geq k_0$.*

Proof By the definition of a stratification of V, we have $\mathsf{V}_{k'} \subseteq \mathsf{V}_{k''}$ for every pair of indices $k', k'' \in \mathbb{N}_0$, $k' < k''$. Hence, for every pseudoinequality $u \leq v$ and for every pair of indices $k', k'' \in \mathbb{N}_0$, $k' < k''$, we obtain the implication

$$\mathsf{V}_{k''} \models u \leq v \;\Rightarrow\; \mathsf{V}_{k'} \models u \leq v.$$

By the definition of a set of the form $\mathcal{P}_{\mathsf{V}_k}[M]$, the preceding implies that

$$\mathcal{P}_{\mathsf{V}_{k''}}[M] \subseteq \mathcal{P}_{\mathsf{V}_{k'}}[M] \subseteq M \times 2^M$$

for all $k', k'' \in \mathbb{N}_0$, $k' < k''$. Since the set $M \times 2^M$ is finite, there must exist an index k_0 such that, for every $k \geq k_0$, we have $\mathcal{P}_{\mathsf{V}_k}[M] = \mathcal{P}_{\mathsf{V}_{k_0}}[M]$. Recall that we have $\mathcal{P}_{\mathsf{V}}[M] = \bigcap_{k=1}^{\infty} \mathcal{P}_{\mathsf{V}_k}[M]$ by Proposition 3.6. Then, for every index $k \geq k_0$, we obtain

$$\mathcal{P}_{\mathsf{V}}[M] = \bigcap_{k=0}^{\infty} \mathcal{P}_{\mathsf{V}_k}[M] = \bigcap_{k=k_0}^{\infty} \mathcal{P}_{\mathsf{V}_k}[M] = \mathcal{P}_{\mathsf{V}_{k_0}}[M] = \mathcal{P}_{\mathsf{V}_k}[M].$$

□

3.3.2 Stratification of Pol V

In what follows, we use the phrase *positive Boolean combination* of sets $L_1, \ldots L_n$. By this, we mean a set that can be created by taking finite unions of finite intersections of $L_1, \ldots, L_n$. Let A be an alphabet and $\mathcal{L} \subseteq 2^{A^*}$ be a set of regular languages. By $\langle \mathcal{L} \rangle_{\text{lattice}}$, we denote the lattice of regular languages generated by $\mathcal{L}$, i.e., the set of all positive Boolean combinations of languages from $\mathcal{L}$.

Let $\mathcal{V}$ be a positive variety of regular languages. For every alphabet A, we define a sequence of sets of regular languages $\{Pol_k(\mathcal{V})(A)\}_{k=0}^{\infty}$. The definition is adopted from [25, Subsection 4.3].

- $Pol_0(\mathcal{V})(A) := \mathcal{V}(A)$,
- $Pol_k(\mathcal{V})(A) := \big\langle Pol_{k-1}(\mathcal{V})(A) \cup \{LaK \mid L, K \in Pol_{k-1}(\mathcal{V})(A), a \in A\}\big\rangle_{\text{lattice}}$ for every $k \geq 1$.

The following lemma comes from the author's PhD thesis [36].

Lemma 3.8 *For every alphabet A, the following relations hold:*

- $Pol_k(\mathcal{V})(A) \subseteq Pol_{k+1}(\mathcal{V})(A)$ *for every $k \in \mathbb{N}_0$,*

- $Pol(\mathcal{V})(A) = \bigcup_{k=0}^{\infty} Pol_k(\mathcal{V})(A)$.

Proof This follows directly from the definitions of sets $Pol_k(\mathcal{V})(A)$ and $Pol(\mathcal{V})(A)$ and the fact that $Pol(\mathcal{V})(A)$ is, as a lattice of regular languages, also closed under intersection. □

The following lemma shows an equivalent definition of $Pol_k(\mathcal{V})(A)$. We will use it in the proof of Lemma 3.12.

Lemma 3.9 *Let* $k \in \mathbb{N}$. *Then*

$$Pol_k(\mathcal{V})(A) = \langle \mathcal{V}(A) \cup \{LaK \mid L, K \in Pol_{k-1}(\mathcal{V})(A), a \in A\} \rangle_{lattice}.$$

Proof The proof goes by induction on k. If $k = 1$, then $Pol_{k-1}(\mathcal{V})(A) = Pol_0(\mathcal{V})(A) = \mathcal{V}(A)$ and the proof is done. Suppose that $k \geq 2$. To shorten the notation, let

$$Q_l := \{LaK \mid L, K \in Pol_l(\mathcal{V})(A), a \in A\}$$

for every $l \in \mathbb{N}_0$ in what follows. Then, using successively the definition of $Pol_k(\mathcal{V})(A)$, the induction assumption, an obvious property of the generation of a lattice, and the inclusion $Q_{k-2} \subseteq Q_{k-1}$ following from Lemma 3.8, we obtain

$$\begin{aligned} Pol_k(\mathcal{V})(A) &= \left\langle Pol_{k-1}(\mathcal{V})(A) \cup Q_{k-1} \right\rangle_{\text{lattice}} = \\ &= \left\langle \langle \mathcal{V}(A) \cup Q_{k-2} \rangle_{\text{lattice}} \cup Q_{k-1} \right\rangle_{\text{lattice}} = \\ &= \left\langle \mathcal{V}(A) \cup Q_{k-2} \cup Q_{k-1} \right\rangle_{\text{lattice}} = \\ &= \left\langle \mathcal{V}(A) \cup Q_{k-1} \right\rangle_{\text{lattice}}. \end{aligned}$$

This is precisely what we needed to prove. □

The following propositions and lemmas come from the author's PhD thesis [36], where only a locally finite variety of regular languages $\mathcal{V}$ was considered. In this paper, they are generalized to an arbitrary positive variety of regular languages $\mathcal{V}$ whenever possible (Proposition 3.10 and Proposition 3.11).

Proposition 3.10 *For every* $k \in \mathbb{N}_0$, $Pol_k(\mathcal{V})$ *is a positive variety of regular languages. Moreover, if* $\mathcal{V}$ *is locally finite, then* $Pol_k(\mathcal{V})$ *is also locally finite.*

Proof The proof is inspired by the proofs of [8, Propositions 2.1.1 and 2.2.1] by Arfi, which show that the polynomial closure of a variety of regular languages is closed under quotients, concatenation, and preimages under homomorphisms. Our proof is very similar. It is even a bit simpler due to the inductive definition of $Pol_k(\mathcal{V})$.

The proof goes by induction on k. If $k = 0$, $Pol_k(\mathcal{V}) = \mathcal{V}$ is a positive variety of regular languages, by the definition of $\mathcal{V}$. It is locally finite if $\mathcal{V}$ is. Suppose that $k \geq 1$. Let A be an alphabet. Directly from the definition of $Pol_k(\mathcal{V})$, the set

$Pol_k(\mathcal{V})(A)$ is obviously a lattice of regular languages. If $\mathcal{V}$ is locally finite, then the lattice $Pol_{k-1}(\mathcal{V})(A)$ is finite by the induction assumption. Then, by the definition, the lattice $Pol_k(\mathcal{V})(A)$ is also finite. Further, we show that it is quotienting.

Let $L \in Pol_k(\mathcal{V})(A)$ be an arbitrary language and a be a letter. We need to show that the languages $a^{-1}L$ and La^{-1} belong to the set $Pol_k(\mathcal{V})(A)$ as well. We show the property $a^{-1}L \in Pol_k(\mathcal{V})(A)$, the proof of the property $La^{-1} \in Pol_k(\mathcal{V})(A)$ is symmetrical.

By the definition of $Pol_k(\mathcal{V})(A)$, the language L is a positive Boolean combination of languages from $Pol_{k-1}(\mathcal{V})(A)$ and languages of the form L_1bL_2, where $L_1, L_2 \in Pol_{k-1}(\mathcal{V})(A)$ and $b \in A$. Since we have

$$a^{-1}(K_1 \cup K_2) = a^{-1}K_1 \cup a^{-1}K_2 \quad \text{and} \quad a^{-1}(K_1 \cap K_2) = a^{-1}K_1 \cap a^{-1}K_2$$

for every pair of languages $K_1, K_2 \subseteq A^*$ and the set $Pol_k(\mathcal{V})(A)$ is closed under both union and intersection, we can assume that

$$L \in Pol_{k-1}(\mathcal{V})(A) \cup \{L_1bL_2 \mid L_1, L_2 \in Pol_{k-1}(\mathcal{V})(A), b \in A\}$$

without loss of generality. If $L \in Pol_{k-1}(\mathcal{V})(A)$, then $a^{-1}L \in Pol_{k-1}(\mathcal{V})(A) \subseteq Pol_k(\mathcal{V})(A)$ by the induction assumption.

Suppose that $L = L_1bL_2$, where $L_1, L_2 \in Pol_{k-1}(\mathcal{V})(A)$ and $b \in A$. Then

$$a^{-1}L = a^{-1}(L_1bL_2) = \begin{cases} (a^{-1}L_1)bL_2 & \text{if } \varepsilon \notin L_1 \text{ or } a \neq b, \\ (a^{-1}L_1)bL_2 \cup L_2 & \text{if } \varepsilon \in L_1 \text{ and } a = b. \end{cases}$$

Since we have $a^{-1}L_1 \in Pol_{k-1}(\mathcal{V})(A)$ by the induction assumption, we obtain that the languages $(a^{-1}L_1)bL_2$ and $(a^{-1}L_1)bL_2 \cup L_2$ belong to the set $Pol_k(\mathcal{V})(A)$. This means that the language $a^{-1}L$ belongs to the set $Pol_k(\mathcal{V})(A)$, as required.

It remains to show that the class $Pol_k(\mathcal{V})$ is closed under preimages under homomorphisms of the form $A^* \to B^*$, where A, B are alphabets.

Let A and B be alphabets and $\varphi\colon A^* \to B^*$ be an arbitrary homomorphism. Let $L \in Pol_k(\mathcal{V})(B) \subseteq B^*$ be an arbitrary language. We need to show that the language $\varphi^{-1}(L)$ belongs to the set $Pol_k(\mathcal{V})(A)$. By the definition of $Pol_k(\mathcal{V})(B)$, the language L is a positive Boolean combination of languages from $Pol_{k-1}(\mathcal{V})(B)$ and languages of the form L_1bL_2, where $L_1, L_2 \in Pol_{k-1}(\mathcal{V})(B)$ and $b \in B$. Since we have

$$\varphi^{-1}(K_1 \cup K_2) = \varphi^{-1}(K_1) \cup \varphi^{-1}(K_2) \quad \text{and} \quad \varphi^{-1}(K_1 \cap K_2) = \varphi^{-1}(K_1) \cap \varphi^{-1}(K_2)$$

for every pair of languages $K_1, K_2 \subseteq B^*$ and the set $Pol_k(\mathcal{V})(A)$ is closed under both union and intersection, we can assume that

$$L \in Pol_{k-1}(\mathcal{V})(B) \cup \{L_1bL_2 \mid L_1, L_2 \in Pol_{k-1}(\mathcal{V})(B), b \in B\}$$

without loss of generality. If $L \in Pol_{k-1}(\mathcal{V})(B)$, then $\varphi^{-1}(L) \in Pol_{k-1}(\mathcal{V})(A) \subseteq Pol_k(\mathcal{V})(A)$ by the induction assumption.

Suppose that $L = L_1 b L_2$, where $L_1, L_2 \in Pol_{k-1}(\mathcal{V})(B)$ and $b \in B$. Let

$$X := \{(u, a, v) \in B^* \times A \times B^* \mid \varphi(a) = ubv\}.$$

Since the alphabet A is finite and, for every $a \in A$, the image $\varphi(a)$ is a finite word, the set X is finite. Then we obtain

$$\varphi^{-1}(L_1 b L_2) = \bigcup_{(u,a,v)\in X} \varphi^{-1}(L_1 u^{-1}) \cdot a \cdot \varphi^{-1}(v^{-1} L_2).$$

Since the set $Pol_{k-1}(\mathcal{V})(B)$ is closed under quotients, we have that $L_1 u^{-1}, v^{-1} L_2 \in Pol_{k-1}(\mathcal{V})(B)$ for every triple $(u, a, v) \in X$. Then, since the class $Pol_{k-1}(\mathcal{V})$ is a positive variety of regular languages by the induction assumption, we obtain that $\varphi^{-1}(L_1 u^{-1}), \varphi^{-1}(v^{-1} L_2) \in Pol_{k-1}(\mathcal{V})(A)$. This implies that, for every $(u, a, v) \in X$,

$$\varphi^{-1}(L_1 u^{-1}) \cdot a \cdot \varphi^{-1}(v^{-1} L_2) \in Pol_k(\mathcal{V})(A).$$

Finally, since the set $Pol_k(\mathcal{V})(A)$ is closed under union, we obtain that the language

$$\varphi^{-1}(L) = \varphi^{-1}(L_1 b L_2) = \bigcup_{(u,a,v)\in X} \varphi^{-1}(L_1 u^{-1}) \cdot a \cdot \varphi^{-1}(v^{-1} L_2)$$

belongs to the set $Pol_k(\mathcal{V})(A)$, as required. □

Let V be a pseudovariety of ordered monoids and $\mathcal{V}$ be the corresponding positive variety of regular languages. For every $k \in \mathbb{N}_0$, let $\mathrm{Pol}_k \mathsf{V}$ be the pseudovariety of ordered monoids corresponding to the positive variety of regular languages $Pol_k(\mathcal{V})$. We obtain the following proposition.

Proposition 3.11 *The sequence* $\{\mathrm{Pol}_k \mathsf{V}\}_{k=0}^{\infty}$ *constitutes a stratification of the pseudovariety* $\mathrm{Pol}\,\mathsf{V}$. *Moreover, if* V *is locally finite, then, for every* $k \in \mathbb{N}_0$, $\mathrm{Pol}_k \mathsf{V}$ *is also locally finite.*

Proof This follows directly from the definition of a stratification of a pseudovariety and from Proposition 3.10 and Lemma 3.8. □

Now we state lemmas describing how we can factorize (pseudo)inequalities valid in a pseudovariety $\mathrm{Pol}_k \mathsf{V}$, where the pseudovariety V is *locally finite*.

The following lemma comes from the paper [25][7]. It provides a nice characterization of inequalities valid in $\mathrm{Pol}_k \mathsf{V}$. Its proof here is a bit different from the proof in [25].

Lemma 3.12 ([25, Lemma 4.9]). *Let* V *be a locally finite pseudovariety. Let* $u, v \in A^*$, $k \in \mathbb{N}$. *Then* $\mathrm{Pol}_k \mathsf{V} \models u \le v$ *if and only if the following two conditions hold:*

1. *The relation* $\mathsf{V} \models u \le v$ *holds.*
2. *For every factorization* $u = u_1 a u_2$*, where* $u_1, u_2 \in A^*$*,* $a \in A$*, there exist words* $v_1, v_2 \in A^*$ *such that*

[7] More precisely, it is a translation of the corresponding statement from [25] into terms of pseudovarieties.

- $v = v_1 a v_2$,
- $\mathrm{Pol}_{k-1}\,\mathsf{V} \models u_i \leq v_i$ *for* $i = 1, 2$.

Proof Let $\mathrm{Pol}_k\,\mathsf{V} \models u \leq v$. Since we have $\mathsf{V} \subseteq \mathrm{Pol}_k\,\mathsf{V}$, we obtain $\mathsf{V} \models u \leq v$. Let $u = u_1 a u_2$, where $u_1, u_2 \in A^*$, $a \in A$, be an arbitrary factorization. For $i = 1, 2$, define

$$L_i = \bigcap_{\substack{L \in Pol_{k-1}(\mathcal{V})(A),\\ u_i \in L}} L.$$

Then $L_1, L_2 \in Pol_{k-1}(\mathcal{V})(A)$, hence $L_1 a L_2 \in Pol_k(\mathcal{V})(A)$. From the properties $\mathrm{Pol}_k\,\mathsf{V} \models u_1 a u_2 \leq v$ and $u_1 a u_2 \in L_1 a L_2$, we obtain $v \in L_1 a L_2$. Hence there exist words $v_1 \in L_1$, $v_2 \in L_2$ such that $v = v_1 a v_2$. Let $i \in \{1, 2\}$. Let L be an arbitrary language from $Pol_{k-1}(\mathcal{V})(A)$ satisfying $u_i \in L$. Then $v_i \in L_i \subseteq L$. This implies $\mathrm{Pol}_{k-1}\,\mathsf{V} \models u_i \leq v_i$, as required.

It remains to prove the opposite implication. Suppose that the relation $\mathsf{V} \models u \leq v$ holds and, for every factorization $u = u_1 a u_2$, where $u_1, u_2 \in A^*$, $a \in A$, there exist words $v_1, v_2 \in A^*$ such that $v = v_1 a v_2$ and $\mathrm{Pol}_{k-1}\,\mathsf{V} \models u_i \leq v_i$ for $i = 1, 2$. We prove that the relation $\mathrm{Pol}_k\,\mathsf{V} \models u \leq v$ holds.

Let $L \in Pol_k(\mathcal{V})(A)$ be an arbitrary language such that $u \in L$. We need to show that $v \in L$. By Lemma 3.9, L is a positive Boolean combination of languages K of the two following forms:

1. $K \in \mathcal{V}(A)$,
2. there exist languages $K_1, K_2 \in Pol_{k-1}(\mathcal{V})(A)$ and a letter $a \in A$ such that $K = K_1 a K_2$.

Let K be such a language and assume that $u \in K$. We prove that also $v \in K$. In the first case when $K \in \mathcal{V}(A)$, using the relation $\mathsf{V} \models u \leq v$ from the assumption, we obtain directly that $v \in K$. It remains to consider the second case. Since $u \in K_1 a K_2$, there exists words $u_1 \in K_1$, $u_2 \in K_2$ such that $u = u_1 a u_2$. Then, by the assumption, there exist words $v_1, v_2 \in A^*$ such that $v = v_1 a v_2$ and $\mathrm{Pol}_{k-1}\,\mathsf{V} \models u_i \leq v_i$ for $i = 1, 2$. Let $i \in \{1, 2\}$. From the relation $\mathrm{Pol}_{k-1}\,\mathsf{V} \models u_i \leq v_i$, we obtain that $v_i \in K_i$. This implies that $v = v_1 a v_2 \in K_1 \cdot a \cdot K_2 = K$, as required.

Finally, since $u \in L$ and L is a positive Boolean combination of languages K of the forms 1 and 2, we obtain that also $v \in L$, which completes the proof. □

As a corollary of Lemma 3.12, we obtain the following lemma, which is convenient for the application in proofs. Both its formulation and the proof are inspired by an analogous statement from the author's master's thesis [9, Corollary 3.45], where inequalities of words defined by use of the first-order logic on words have been considered.

Lemma 3.13 *Let* V *be a locally finite pseudovariety. Let* $u_1, \ldots, u_n, v \in A^*$, *where* $n \in \mathbb{N}$, *be arbitrary words. Let* k *be an integer satisfying* $k \geq n - 1$. *Let* $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \ldots u_n \leq v$. *Then there exist words* $v_1, \ldots, v_n \in A^*$ *such that*

- $v = v_1 \ldots v_n$,
- $\mathrm{Pol}_{k-(n-1)}\,\mathsf{V} \models u_i \leq v_i$ *for* $i = 1, \ldots, n$.

Proof By induction on n. For $n = 1$, it is obvious. Let $n = 2$. By the assumption, we have $\mathrm{Pol}_k \mathsf{V} \models u_1 u_2 \le v$. If $u_2 = \varepsilon$, it suffices to choose $v_1 = v$, $v_2 = \varepsilon$. If $|u_2| \ge 1$, there exist $a \in A$, $u_2' \in A^*$ such that $u_2 = a u_2'$. Then we have $u = u_1 a u_2'$. By Lemma 3.12, there exist words $v_1, v_2' \in A^*$ such that $v = v_1 a v_2'$, $\mathrm{Pol}_{k-1} \mathsf{V} \models u_1 \le v_1$, $\mathrm{Pol}_{k-1} \mathsf{V} \models u_2' \le v_2'$. Choose $v_2 = a v_2'$. Then we obtain $v = v_1 v_2$, $\mathrm{Pol}_{k-1} \mathsf{V} \models u_2 = a u_2' \le a v_2' = v_2$.

Let $n > 2$. By the assumption, we have $\mathrm{Pol}_k \mathsf{V} \models u_1 \dots u_n \le v$. Denote $u_1' := u_1 \dots u_{n-1}$. Then $\mathrm{Pol}_k \mathsf{V} \models u_1' u_n \le v$. By the induction assumption, there exist words $v_1', v_n \in A^*$ such that $v = v_1' v_n$, $\mathrm{Pol}_{k-1} \mathsf{V} \models u_1' \le v_1'$, $\mathrm{Pol}_{k-1} \mathsf{V} \models u_n \le v_n$. From the property $\mathrm{Pol}_{k-1} \mathsf{V} \models u_1 \dots u_{n-1} = u_1' \le v_1'$, using the induction assumption, we obtain words $v_1, \dots, v_{n-1} \in A^*$ satisfying $v_1' = v_1 \dots v_{n-1}$, $\mathrm{Pol}_{k-1-(n-2)} \mathsf{V} \models u_i \le v_i$ for $i = 1, \dots, n-1$. Since we have $v = v_1' v_n = v_1 \dots v_{n-1} v_n$ and $k-1-(n-2) = k-(n-1)$, the lemma has been proven. □

3.3.3 Stratification of Pol(Pol V)$^{\mathrm{d}}$

At first, recall that we are interested in a pseudovariety of the form $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ since it corresponds to level 3/2 of a concatenation hierarchy with $\mathsf{V}_0 = \mathsf{V}$.

Let V be a pseudovariety. If we denote $\mathsf{U} := (\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, we can stratify the pseudovariety $\mathrm{Pol}\,\mathsf{U}$ in the way described in Subsection 3.3.2. However, this is not the stratification of $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ we are going to use. In our proof of the ω-reducibility, we need to work with a stratification into *locally finite* pseudovarieties. But, even for a locally finite pseudovariety V, the pseudovariety $\mathsf{U} = (\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ does not have to be locally finite and so the pseudovarieties $\mathrm{Pol}_k \mathsf{U}$ do not have to be locally finite either.

Instead, we use the following special “double stratification” of $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. For every $k \in \mathbb{N}_0$, let

$$\mathsf{U}_k := (\mathrm{Pol}_k \mathsf{V})^{\mathrm{d}}.$$

In the first step, we consider the sequence of pseudovarieties $\{\mathrm{Pol}\,\mathsf{U}_k\}_{k=0}^{\infty}$. Note that the pseudovarieties $\mathrm{Pol}\,\mathsf{U}_k$ also do not have to be locally finite. However, we can further stratify each of these pseudovarieties into a sequence of *locally finite* pseudovarieties in the way described in Subsection 3.3.2. Also the following propositions and corollary come from the author’s PhD thesis [36].

Proposition 3.14 *The sequence* $\{\mathrm{Pol}\,\mathsf{U}_k\}_{k=0}^{\infty}$ *constitutes a stratification of the pseudovariety* $\mathrm{Pol}\,\mathsf{U} = \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

Proof By the definition of a stratification, we need to prove that the following two conditions are satisfied:

(i) $\mathrm{Pol}\,\mathsf{U}_k \subseteq \mathrm{Pol}\,\mathsf{U}_{k+1}$ for every $k \in \mathbb{N}_0$,
(ii) $\bigcup_{k=0}^{\infty} \mathrm{Pol}\,\mathsf{U}_k = \mathrm{Pol}\,\mathsf{U}$.

Let $k \in \mathbb{N}_0$ be an arbitrary index. Since we have $\mathrm{Pol}_k \mathsf{V} \subseteq \mathrm{Pol}_{k+1} \mathsf{V}$, we obtain $(\mathrm{Pol}_k \mathsf{V})^{\mathrm{d}} \subseteq (\mathrm{Pol}_{k+1} \mathsf{V})^{\mathrm{d}}$. This implies that

$$\mathrm{Pol}\,\mathsf{U}_k = \mathrm{Pol}(\mathrm{Pol}_k \mathsf{V})^{\mathrm{d}} \subseteq \mathrm{Pol}(\mathrm{Pol}_{k+1} \mathsf{V})^{\mathrm{d}} = \mathrm{Pol}\,\mathsf{U}_{k+1},$$

as required.

It remains to show that Condition (ii) is satisfied. Let $\mathcal{U}$ be the positive variety of regular languages corresponding to U. Similarly, let $\mathcal{U}_k$ be the positive variety of regular languages corresponding to U_k. Then we have $\mathcal{U} = \text{Co-}Pol(\mathcal{V})$ and $\mathcal{U}_k = \text{Co-}Pol_k(\mathcal{V})$. Using the definitions of classes of the form Co-$\mathcal{V}$ and $Pol(\mathcal{V})$, we obtain

$$Pol(\mathcal{U})(A) = Pol\big(\text{Co-}Pol(\mathcal{V})\big)(A) = Pol\big(\text{Co-}\bigcup_{k=0}^{\infty} Pol_k(\mathcal{V})\big)(A) =$$
$$= Pol\big(\bigcup_{k=0}^{\infty} \text{Co-}Pol_k(\mathcal{V})\big)(A) = \bigcup_{k=0}^{\infty} Pol\big(\text{Co-}Pol_k(\mathcal{V})\big)(A) = \bigcup_{k=0}^{\infty} Pol\big(\mathcal{U}_k\big)(A).$$

This is equivalent to the relation

$$\mathrm{Pol}\,\mathsf{U} = \bigcup_{k=0}^{\infty} \mathrm{Pol}\,\mathsf{U}_k.$$

Altogether, we have proven that $\{\mathrm{Pol}\,\mathsf{U}_k\}_{k=0}^{\infty}$ is a stratification of the pseudovariety $\mathrm{Pol}\,\mathsf{U}$. □

In the second step, we consider, for every $k \in \mathbb{N}_0$, the sequence of pseudovarieties $\{\mathrm{Pol}_l\,\mathsf{U}_k\}_{l=0}^{\infty}$.

Proposition 3.15 *Let* V *be a locally finite pseudovariety and* $k \in \mathbb{N}_0$ *be an arbitrary index. The sequence* $\{\mathrm{Pol}_l\,\mathsf{U}_k\}_{l=0}^{\infty}$*, where* $\mathsf{U}_k = (\mathrm{Pol}_k\,\mathsf{V})^{\mathrm{d}}$*, constitutes a stratification of the pseudovariety* $\mathrm{Pol}\,\mathsf{U}_k$ *into locally finite pseudovarieties.*

Proof Since the pseudovariety V is locally finite, the pseudovariety $\mathrm{Pol}_k\,\mathsf{V}$ is also locally finite by Proposition 3.11. Hence also the pseudovariety $\mathsf{U}_k = (\mathrm{Pol}_k\,\mathsf{V})^{\mathrm{d}}$ is locally finite. Using Proposition 3.11 again, we obtain that the sequence of pseudovarieties $\{\mathrm{Pol}_l\,\mathsf{U}_k\}_{l=0}^{\infty}$ constitutes a stratification of the pseudovariety $\mathrm{Pol}\,\mathsf{U}_k$ into locally finite pseudovarieties. □

Corollary 3.16

$$\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}} = \bigcup_{k=0}^{\infty}\bigcup_{l=0}^{\infty} \mathrm{Pol}_l(\mathrm{Pol}_k\,\mathsf{V})^{\mathrm{d}}. \tag{3}$$

Proof By Propositions 3.14 and 3.15, we have

$$\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}} = \bigcup_{k=0}^{\infty} \mathrm{Pol}(\mathrm{Pol}_k\,\mathsf{V})^{\mathrm{d}} \quad \text{and} \quad \mathrm{Pol}(\mathrm{Pol}_k\,\mathsf{V})^{\mathrm{d}} = \bigcup_{l=0}^{\infty} \mathrm{Pol}_l(\mathrm{Pol}_k\,\mathsf{V})^{\mathrm{d}}.$$

Altogether, we obtain Relation (3). □

4 ω-reducibility of $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$

In this section, we prove the ω-reducibility of sets of the form $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$, where the pseudovariety V is locally finite. Let V be a locally finite pseudovariety. To prove the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$, we use the stratification of $\mathrm{Pol}\,\mathsf{V}$ from Subsection 3.3.2 and Proposition 3.4 for the locally finite pseudovariety $\mathsf{W} = \mathrm{Pol}_k\,\mathsf{V}$, where k is a specific number such that $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M] = \mathcal{P}_{\mathrm{Pol}_k\,\mathsf{V}}[M]$.

Let U be a pseudovariety of ordered monoids. Let u be a pseudoword and U be a set of pseudowords. Recall the notation

$$\mathsf{U} \models u \leq U$$

for a system of inequalities

$$\left\{\mathsf{U} \models u \leq w \mid w \in U\right\}$$

introduced in Subsection 3.2.

In the first subsection, we define a *k-index* of a sequence of words, which we use in the next subsection. We also state its basic properties.

In the second subsection, we prove an important lemma, which describes a special way of a "factorization" of inequalities of the form $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \dots u_n \leq U$, where n is arbitrarily large, $u_1, \dots, u_n \in A^*$ and $U \subseteq A^*$. This lemma is crucial for our proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ by a factorization forest.

Then in the third subsection, we show how to prove the ω-reducibility for $\mathrm{Pol}\,\mathsf{V}$ of a pair $\big(\alpha(u_1 \dots u_n), \alpha(U)\big) \in \mathcal{P}_{\mathrm{Pol}_k\,\mathsf{V}}[M]$, where $\alpha\colon A^* \to M$ is a surjective homomorphism and $\alpha(u_1) = \dots = \alpha(u_n) = e \in E(M)$, using the ω-reducibility of its special "factors" created by the lemma mentioned above. Moreover, by addition of an extra condition, we are able to extend this process to the ω-reducibility for $\mathrm{Pol}\,\mathsf{U}$, where a pseudovariety U is not required to be locally finite. This generalization will be useful for the proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ in Section 5.

Finally, in the fourth subsection, we perform the proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$, using the results of the third subsection and the induction on the height of words in a factorization forest.

In the last (fifth) subsection, we describe a connection between our proof of the ω-reducibility and the algorithm computing the set $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ from [25].

Now fix a *locally finite* pseudovariety of ordered monoids V, an alphabet A, a finite monoid M, and a surjective homomorphism $\alpha\colon A^* \to M$.

4.1 *k*-index of a sequence of words

Most of the material in this subsection is adopted from the author's PhD thesis [36, Subsection 3.2.1].

The following definition of a *k-index* of a sequence of words is inspired by the definitions of an "*index of an (e, p)-decomposition*" from the paper [28, p. 39] and an "*index of a sequence*" from [25, p. 29] used there in a similar context.

Definition 4.1 Let $n, k \in \mathbb{N}$. Let $(u_1, \dots, u_n) \in (A^*)^n$ be a sequence of words. For $i = 1, \dots, n$, let U_i be the maximal subset of A^* such that $\mathrm{Pol}_k\,\mathsf{V} \models u_i \le U_i$. A *$k$-index of the sequence* $(u_1, \dots, u_n)$ is denoted by $\mathrm{i}_k(u_1, \dots, u_n)$ and is defined in the following way:

$$\begin{aligned}\mathrm{i}_k(u_1, \dots, u_n) := \Big|\big\{E \in E(2^M) \mid \exists \iota, \kappa, \lambda \in \{1, \dots, n\}, \iota \le \kappa < \lambda, \kappa - \iota < 2^{|M|} :\\ \alpha(U_\iota \cdots U_\kappa) \cdot E = \alpha(U_\iota \cdots U_\kappa), E = \big(\alpha(U_{\kappa+1} \cdots U_\lambda)\big)^\omega\big\}\Big|.\end{aligned}$$

In other words, the k-index of a sequence $(u_1, \dots, u_n)$ counts a number of different idempotents of 2^M of the form $\big(\alpha(U_{\kappa+1} \cdots U_\lambda)\big)^\omega$ that can be inserted into the sequence $\alpha(U_1), \dots, \alpha(U_n)$ at a position κ without changing the product $\alpha(U_\iota) \cdots \alpha(U_\kappa)$ for some index $\iota \in \{\kappa - 2^{|M|} + 1, \dots, \kappa\}$.

Note that $\mathrm{i}_k(u_1, \dots, u_n) \in \{0, \dots, 2^{|M|}\}$.

The following lemma is standard in the theory of finite semigroups.

Lemma 4.2 *Let M be a finite monoid. Let $n > |M|$, $s_1, \dots, s_n \in M$. Then there exist indices $\kappa, \lambda \in \{1, \dots, |M| + 1\}$, $\kappa < \lambda$ such that*

$$s_1 \dots s_\kappa = s_1 \dots s_\kappa \, (s_{\kappa+1} \dots s_\lambda)^\omega.$$

Proof Consider the sequence

$$s_1,\ s_1 s_2,\ \dots,\ s_1 s_2 \dots s_{|M|} s_{|M|+1}$$

of elements of the monoid M. By the pigeonhole principle, there must exist indices $1 \le \kappa < \lambda \le |M| + 1$ such that $s_1 \dots s_\kappa = s_1 \dots s_\kappa s_{\kappa+1} \dots s_\lambda$. This equality implies

$$\begin{aligned}s_1 \dots s_\kappa &= (s_1 \dots s_\kappa) \cdot (s_{\kappa+1} \dots s_\lambda) = (s_1 \dots s_\kappa) \cdot (s_{\kappa+1} \dots s_\lambda)^2 = \\ &= \dots = (s_1 \dots s_\kappa) \cdot (s_{\kappa+1} \dots s_\lambda)^\omega.\end{aligned}$$

□

Corollary 4.3 *Let $n > 2^{|M|}$ and $u_1, \dots, u_n \in A^*$. Then $\mathrm{i}_k(u_1, \dots, u_n) \ge 1$.*

Proof For every $i \in \{1, \dots, n\}$, we have $\alpha(U_i) \in 2^M$. Since the monoid 2^M has size $2^{|M|}$, by Lemma 4.2, there exist indices $\kappa, \lambda \in \{1, \dots, 2^{|M|} + 1\}$, $\kappa < \lambda$ such that

$$\alpha(U_1 \cdots U_\kappa) = \alpha(U_1 \cdots U_\kappa) \cdot \big(\alpha(U_{\kappa+1} \cdots U_\lambda)\big)^\omega.$$

We have $\kappa - 1 < 2^{|M|}$. By the definition of k-index, we obtain that

$$\mathrm{i}_k(u_1, \dots, u_n) \ge 1.$$

□

4.2 Factorization of inequalities

The following lemma generalizes Lemma 3.13 on the factorization of inequalities valid in a locally finite pseudovariety $\mathrm{Pol}_k\,\mathsf{V}$.

Lemma 4.4 *Let $n, k \in \mathbb{N}$, $k \geq n-1$, $u_1, \ldots, u_n \in A^*$, $U \subseteq A^*$. Let $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \ldots u_n \leq U$. For $i = 1, \ldots, n$, let $m_i \in \{n-1, \ldots, k\}$ and let U_i be the maximal subset of A^* such that $\mathrm{Pol}_{k-m_i}\,\mathsf{V} \models u_i \leq U_i$. Then $U \subseteq U_1 \cdots U_n$.*

Proof Let $v \in U$ be an arbitrary word. Since $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \ldots u_n \leq v$ by the assumption, there exist words $v_1, \ldots, v_n \in A^*$ satisfying $v = v_1 \ldots v_n$ and $\mathrm{Pol}_{k-(n-1)}\,\mathsf{V} \models u_i \leq v_i$ for $i = 1, \ldots, n$ by Lemma 3.13. Since we have $m_i \geq n-1$, i.e., $k-(n-1) \geq k-m_i$, the preceding relation implies that also $\mathrm{Pol}_{k-m_i}\,\mathsf{V} \models u_i \leq v_i$ for $i = 1, \ldots, n$. By the definition of the sets U_i, we obtain that $v_i \in U_i$ for $i = 1, \ldots, n$, i.e., $v = v_1 \ldots v_n \in U_1 \cdots U_n$. Since $v \in U$ was chosen arbitrarily, we obtain $U \subseteq U_1 \cdots U_n$. □

The following lemma is crucial for our proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$. It shows a way how to "factorize" inequalities of the form $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \ldots u_n \leq U$ in order to be able to prove the ω-reducibility of a pair $\big(\alpha(u_1 \ldots u_n), \alpha(U)\big) \in \mathcal{P}_{\mathrm{Pol}_k\,\mathsf{V}}[M]$, where $\alpha(u_1) = \cdots = \alpha(u_n) = e \in E(M)$, inductively, using this factorization. We will see its application in the next subsection.

The lemma is similar to [25, Lemma 6.2], which describes *covering* of a specific regular language $L \subseteq \alpha^{-1}(e)$, where $e \in E(M)$.

Lemma 4.5 *Let $n \in \mathbb{N}$. Let $u_1, \ldots, u_n \in A^*$ be arbitrary words. Let $k \in \mathbb{N}$, $k \geq 2^{|M|}$. Let U be a subset of A^* such that*

$$\mathrm{Pol}_{k+2\cdot \mathrm{i}_{k-2^{|M|}}(u_1,\ldots,u_n)}\,\mathsf{V} \models u_1 \ldots u_n \leq U.$$

For $i = 1, \ldots, n$, let U_i be the maximal subset of A^* satisfying

$$\mathrm{Pol}_{k-2^{|M|}}\,\mathsf{V} \models u_i \leq U_i.$$

Then there exist sets $W_1, \ldots, W_m \subseteq A^*$, where $m \in \mathbb{N}$, satisfying the following two conditions:

1. *$U \subseteq W_1 \cdots W_m$,*
2. *for every $i \in \{1, \ldots, m\}$, the set W_i is of the form*
   - *(i) $W_i = U_\iota \cdot U_{\iota+1} \cdots U_\kappa$ for some indices $\iota, \kappa \in \{1, \ldots, n\}$, $\iota \leq \kappa$, or*
   - *(ii) $W_i = U_\iota \cdot U_{\iota+1} \cdots U_\kappa \cdot \overline{U} \cdot U_{\iota'} \cdot U_{\iota'+1} \cdots U_{\kappa'}$ for some set $\overline{U} \subseteq A^*$ such that*

$$\mathrm{Pol}_{k-2^{|M|}}\,\mathsf{V} \models u_{\kappa+1} \ldots u_{\iota'-1} \leq \overline{U},$$

for some indices $\iota, \kappa, \iota', \kappa' \in \{1, \ldots, n\}$, $\iota \leq \kappa < \iota' \leq \kappa'$ for which there exists an idempotent $E \in E(2^M)$ satisfying the relations

$$\alpha(U_\iota \cdots U_\kappa) = \alpha(U_\iota \cdots U_\kappa) \cdot E, \quad \alpha(U_{\iota'} \cdots U_{\kappa'}) = \alpha(U_{\iota'} \cdots U_{\kappa'}) \cdot E,$$

and

$$E = \left(\alpha(U_{\kappa+1} \cdots U_{\lambda})\right)^K$$

for some index $\lambda \in \{\kappa + 1, \ldots, n\}$*,* $\lambda - \kappa \leq 2^{|M|}$*, and a number* $K \in \{1, \ldots, 2^{|M|}\}$*.*

Remark 4.6 According to the formulation of Lemma 4.5, it is possible that some of the sets $W_1, \ldots, W_m$ are of the form (ii), where $\iota' = \kappa + 1$. In this case, by the notation $u_{\kappa+1} \ldots u_{\iota'-1}$ we mean the empty word ε.

Proof of Lemma 4.5 The proof goes by induction on the index $\mathrm{i}_{k-2^{|M|}}(u_1, \ldots, u_n)$. To simplify the notation, we will write $\mathrm{i}(u_1, \ldots, u_n)$ in place of $\mathrm{i}_{k-2^{|M|}}(u_1, \ldots, u_n)$ throughout this proof.

If $\mathrm{i}(u_1, \ldots, u_n) = 0$, then $n \leq 2^{|M|}$ by Corollary 4.3. By the assumption, we have $\mathrm{Pol}_k \mathsf{V} \models u_1 \ldots u_n \leq U$. Using Lemma 4.4, we obtain that $U \subseteq U_1 \cdots U_n$. It suffices to choose $W_1 := U_1 \cdots U_n$.

Now suppose that $\mathrm{i}(u_1, \ldots, u_n) \geq 1$. If $n \leq 2^{|M|}$, we proceed in the same way as in the previous case. Suppose that $n > 2^{|M|}$. Then, by Lemma 4.2, there exist indices $1 \leq \kappa < \lambda \leq 2^{|M|} + 1$ such that $\alpha(U_1 \cdots U_\kappa) = \alpha(U_1 \cdots U_\kappa) \cdot \left(\alpha(U_{\kappa+1} \cdots U_\lambda)\right)^\omega$. Let

$$E := \left(\alpha(U_{\kappa+1} \cdots U_\lambda)\right)^\omega.$$

Since the monoid 2^M has size $2^{|M|}$, there exists an integer $K \in \{1, \ldots, 2^{|M|}\}$ such that

$$E = \left(\alpha(U_{\kappa+1} \cdots U_\lambda)\right)^\omega = \left(\alpha(U_{\kappa+1} \cdots U_\lambda)\right)^K.$$

Recall that we have the relations

$$\mathrm{Pol}_{k+2\cdot \mathrm{i}(u_1,\ldots,u_n)} \mathsf{V} \models u_1 \ldots u_n \leq U$$

and

$$U_i = \{v_i \in A^* \mid \mathrm{Pol}_{k-2^{|M|}} \mathsf{V} \models u_i \leq v_i\} \quad \text{for } i = 1, \ldots, n$$

by the assumption. Let

$$W_0 := \{w_0 \in A^* \mid \mathrm{Pol}_{k-2^{|M|}+\kappa-1} \mathsf{V} \models u_1 \ldots u_\kappa \leq w_0\}.$$

By Lemma 4.4, we obtain that $W_0 \subseteq U_1 \cdots U_\kappa$. Further, we have

$$k - 2^{|M|} + \kappa - 1 \leq k - 2^{|M|} + 2^{|M|} - 1 = k - 1 < k + 2 \cdot \mathrm{i}(u_1, \ldots, u_n) - 1.$$

Let

$$W' := \{w' \in A^* \mid \mathrm{Pol}_{k+2\cdot \mathrm{i}(u_1,\ldots,u_n)-1} \models u_{\kappa+1} \ldots u_n \leq w'\}.$$

Then, by Lemma 4.4, we obtain $U \subseteq W_0 W' \subseteq U_1 \cdots U_\kappa \cdot W'$.

If $\mathrm{i}(u_{\kappa+1}, \ldots, u_n) < \mathrm{i}(u_1, \ldots, u_n)$, since

$$\begin{aligned} k + 2 \cdot \mathrm{i}(u_{\kappa+1}, \ldots, u_n) &\leq k + 2 \cdot \left(\mathrm{i}(u_1, \ldots, u_n) - 1\right) = \\ &= k + 2 \cdot \mathrm{i}(u_1, \ldots, u_n) - 2 < \end{aligned}$$

$$< k + 2 \cdot \mathrm{i}(u_1, \ldots, u_n) - 1,$$

we obtain the relation

$$\mathrm{Pol}_{k+2\cdot \mathrm{i}(u_{\kappa+1},\ldots,u_n)} \models u_{\kappa+1} \ldots u_n \leq W'.$$

Then, by the induction assumption, there exist sets $W'_1, \ldots, W'_{m'} \subseteq A^*$, where $m' \in \mathbb{N}$, satisfying the following two conditions:

1. $W' \subseteq W'_1 \cdots W'_{m'}$,
2. for every $i \in \{1, \ldots, m'\}$, the set W'_i is of the form (i) or (ii).

If we choose $W_1 = U_1 \cdots U_\kappa$, we obtain $U \subseteq W_1 \cdot W'_1 \cdots W'_{m'}$.

Now suppose that $\mathrm{i}(u_{\kappa+1}, \ldots, u_n) = \mathrm{i}(u_1, \ldots, u_n)$. Then, by the definition of $\mathrm{i}(u_{\kappa+1}, \ldots, u_n)$, there exist indices ι', κ', $\kappa + 1 \leq \iota' \leq \kappa' < n$, $\kappa' - \iota' < 2^{|M|}$ such that

$$\alpha(U_{\iota'} \cdots U_{\kappa'}) = \alpha(U_{\iota'} \cdots U_{\kappa'}) \cdot E.$$

Choose the biggest possible such ι' and some κ' for this ι'. Then we have

$$\mathrm{i}(u_{\kappa'+1}, \ldots, u_n) < \mathrm{i}(u_1, \ldots, u_n).$$

Hence

$$\begin{aligned} k + 2 \cdot \mathrm{i}(u_{\kappa'+1}, \ldots, u_n) \leq k + 2 \cdot \big(\mathrm{i}(u_1, \ldots, u_n) - 1\big) = \\ = k + 2 \cdot \mathrm{i}(u_1, \ldots, u_n) - 2. \end{aligned}$$

Recall that we have

$$\mathrm{Pol}_{k+2\cdot \mathrm{i}(u_1,\ldots,u_n)-1} \mathsf{V} \models u_{\kappa+1} \ldots u_n \leq W'.$$

Let

$$\overline{W} := \{\overline{w} \in A^* \mid \mathrm{Pol}_{k+2\cdot \mathrm{i}(u_1,\ldots,u_n)-2} \mathsf{V} \models u_{\kappa+1} \ldots u_{\kappa'} \leq \overline{w}\},$$

$$W'' := \{w'' \in A^* \mid \mathrm{Pol}_{k+2\cdot \mathrm{i}(u_{\kappa'+1},\ldots,u_n)} \mathsf{V} \models u_{\kappa'+1} \ldots u_n \leq w''\}.$$

By Lemma 4.4, we obtain that $W' \subseteq \overline{W} W''$. This implies that

$$U \subseteq U_1 \cdots U_\kappa \cdot W' \subseteq U_1 \cdots U_\kappa \cdot \overline{W} \cdot W''.$$

Further, by the induction assumption, there exist sets $W''_1, \ldots, W''_{m''} \subseteq A^*$, where $m'' \in \mathbb{N}$, satisfying the following two conditions:

1. $W'' \subseteq W''_1 \cdots W''_{m''}$,
2. for every $i \in \{1, \ldots, m''\}$, the set W''_i is of the form (i) or (ii).

Then we obtain

$$U \subseteq U_1 \cdots U_\kappa \cdot \overline{W} \cdot W'' \subseteq U_1 \cdots U_\kappa \cdot \overline{W} \cdot W''_1 \cdots W''_{m''}.$$

By the definition of $\overline{W}$, we have

$$\mathrm{Pol}_{k+2\cdot \mathrm{i}(u_1,\ldots,u_n)-2}\,\mathsf{V} \models u_{\kappa+1}\ldots u_{\kappa'} \le \overline{W}.$$

Further, we know that, for every $i \in \{\kappa+1,\ldots,\kappa'\}$,

$$U_i = \{v_i \in A^* \mid \mathrm{Pol}_{k-2^{|M|}}\,\mathsf{V} \models u_i \le v_i\}.$$

Let

$$\overline{U} := \{\overline{u} \in A^* \mid \mathrm{Pol}_{k-2^{|M|}}\,\mathsf{V} \models u_{\kappa+1}\ldots u_{\iota'-1} \le \overline{u}\},$$

where we put $u_{\kappa+1}\ldots u_{\iota'-1} := \varepsilon$ in the case when $\iota' = \kappa+1$ (as in Remark 4.6). Since $\kappa' - \iota' < 2^{|M|}$ and

$$k + 2\cdot \mathrm{i}(u_1,\ldots,u_n) - 2 - 2^{|M|} \ge k + 2 - 2 - 2^{|M|} = k - 2^{|M|},$$

we obtain that $\overline{W} \subseteq \overline{U}\cdot U_{\iota'}\cdots U_{\kappa'}$ by Lemma 4.4.

Finally, if we choose $W_1 := U_1\cdots U_\kappa\cdot\overline{U}\cdot U_{\iota'}\cdots U_{\kappa'}$, we obtain

$$\begin{aligned} U \subseteq U_1\cdots U_\kappa\cdot\overline{W}\cdot W_1''\cdots W_{m''}'' \subseteq U_1\cdots U_\kappa\cdot\overline{U}\cdot U_{\iota'}\cdots U_{\kappa'}\cdot W_1''\cdots W_{m''}'' = \\ = W_1 \cdot W_1''\cdots W_{m''}'', \end{aligned}$$

where W_1 and all the sets W_i'' satisfy Condition 2 of Lemma 4.5. The proof of Lemma 4.5 has been finished. □

4.3 Properties of ω-reducible pairs

Let U be an arbitrary pseudovariety of ordered monoids. Recall that we have fixed an alphabet A, a finite monoid M, and a surjective homomorphism $\alpha\colon A^* \to M$. We say that a pair $(s,S) \in M \times 2^M$ is *ω-reducible for* U if there exists an ω-word $u \in \widehat{\alpha}^{-1}(s)$ such that, for every $t \in S$, there exists an ω-word $v \in \widehat{\alpha}^{-1}(t)$ having the property that $\mathsf{U} \models u \le v$. Then the set $\mathcal{P}_{\mathsf{U}}[M]$ is ω-reducible if and only if every pair $(s,S) \in \mathcal{P}_{\mathsf{U}}[M]$ is ω-reducible for U.

Recall that we have fixed also a locally finite pseudovariety V. In this subsection, we employ Lemma 4.5 to prove the ω-reducibility for $\mathrm{Pol}\,\mathsf{V}$ of pairs $(s,S) \in M\times 2^M$ of the form $(s,S) = \big(e,\alpha(U)\big)$ assuming that the pairs $\big(e,\alpha(U_1)\big),\ldots,\big(e,\alpha(U_n)\big)$ are ω-reducible for $\mathrm{Pol}\,\mathsf{V}$, where $e \in E(M)$, $\alpha(u_1) = \alpha(u_2) = \cdots \alpha(u_n) = e$, and $u_1,u_2,\ldots,u_n \in A^*$ and $U, U_1,\ldots,U_n \subseteq A^*$ are from Lemma 4.5. Moreover, adding an extra condition, we are able to prove the ω-reducibility of these pairs also for $\mathrm{Pol}\,\mathsf{U}$, where U is an *arbitrary* pseudovariety.

4.3.1 Basic properties

We begin with simpler properties of ω-reducible pairs $(s,S) \in M \times 2^M$, which we use in the proof of our target claim.

We will use the following lemma regarding idempotents in finite monoids. An (easy) proof can be found in the author's paper [37].

Lemma 4.7 ([37, Lemma 4.6]). *Let M be a finite monoid, E be an idempotent element of the power monoid 2^M, and $f \in E \subseteq M$ be an arbitrary element. Then there exist elements $f_1, f_3 \in E$, $f_2 \in E \cap E(M)$ such that the equality $f = f_1 f_2 f_3$ holds.*

The following lemmas were proven in [37, Theorem 4.7 and Lemma 4.10] for the case $\mathsf{U} = \operatorname{Pol} \mathsf{W}$, where W is a locally finite selfdual pseudovariety. The proofs for the general case will be analogous. We present the general proofs here, except for the first two (Lemmas 4.8 and 4.9), which are obvious.

Lemma 4.8 *Let U be an arbitrary pseudovariety. Let $s \in M$ be an arbitrary element. The pair $(s, \{s\})$ is ω-reducible for U.*

Lemma 4.9 *Let U be an arbitrary pseudovariety. Let (s, S) be an ω-reducible pair for U and let $S' \subseteq S$. Then the pair (s, S') is also ω-reducible for U.*

Lemma 4.10 *Let U be an arbitrary pseudovariety. Let (s_1, S_1), (s_2, S_2) be ω-reducible pairs for U. Then the pair $(s_1 s_2, S_1 S_2)$ is also ω-reducible for U.*

Proof Suppose that the pairs (s_1, S_1), (s_2, S_2) are ω-reducible for U. This means that there exist ω-words $x_1, x_2 \in \Omega_A^\omega \mathsf{M}$ and sets of ω-words $X_1, X_2 \subseteq \Omega_A^\omega \mathsf{M}$ satisfying the following conditions:

i) $\mathsf{U} \models x_i \leq X_i$ for $i = 1, 2$,
ii) $\widehat{\alpha}(x_i) = s_i, \widehat{\alpha}(X_i) = S_i$ for $i = 1, 2$.

We need to show that the pair $(s_1 s_2, S_1 S_2)$ is also ω-reducible for U, i.e., we need to find an ω-word $x \in \Omega_A^\omega \mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^\omega \mathsf{M}$ satisfying the following conditions:

i) $\mathsf{U} \models x \leq X$,
ii) $\widehat{\alpha}(x) = s_1 s_2, \widehat{\alpha}(X) = S_1 S_2$.

It suffices to choose $x := x_1 x_2$, $X := X_1 X_2$. Indeed, we have

i) $\mathsf{U} \models x = x_1 x_2 \leq X_1 X_2 = X$,
ii) $\widehat{\alpha}(x) = \widehat{\alpha}(x_1)\widehat{\alpha}(x_2) = s_1 s_2,\ \widehat{\alpha}(X) = \widehat{\alpha}(X_1)\widehat{\alpha}(X_2) = S_1 S_2$.

□

Definition 4.11 Let $\mathsf{U}_1, \mathsf{U}_2$ be arbitrary pseudovarieties. We say that a triple $(s, t, T) \in M \times M \times 2^M$ is *ω-reducible for* $(\mathsf{U}_1, \mathsf{U}_2)$ if there exist ω-words $u, v \in \Omega_A^\omega \mathsf{M}$ and a set of ω-words $V \subseteq \Omega_A^\omega \mathsf{M}$ satisfying the following conditions:

i) $\mathsf{U}_1 \models u \leq v, \mathsf{U}_2 \models v \leq V$,
ii) $\widehat{\alpha}(u) = s, \widehat{\alpha}(v) = t, \widehat{\alpha}(V) = T$.

Lemma 4.12 *Let U be an arbitrary pseudovariety. Let $(r, e, E) \in M \times E(M) \times E(2^M)$ be an ω-reducible triple for $(\mathsf{U}^{\mathrm{d}}, \operatorname{Pol} \mathsf{U})$. Then the pair $\big(e, E \cdot \{r, 1\} \cdot E\big)$ is ω-reducible for* $\operatorname{Pol} \mathsf{U}$.

Proof Let $(r, e, E) \in M \times E(M) \times E(2^M)$ be an ω-reducible triple for $\big(\mathsf{U}^{\mathrm{d}}, \operatorname{Pol}\mathsf{U}\big)$. Then there exist ω-words $\overline{u} \in \widehat{\alpha}^{-1}(r)$ and $\overline{v} \in \widehat{\alpha}^{-1}(e)$ satisfying $\mathsf{U}^{\mathrm{d}} \models \overline{u} \leq \overline{v}$ and such that, for every $f \in E$, there exists an ω-word $\overline{w} \in \widehat{\alpha}^{-1}(f)$ satisfying $\operatorname{Pol}\mathsf{U} \models \overline{v} \leq \overline{w}$. To prove the ω-reducibility of the pair $\big(e, E\cdot\{r, 1\}\cdot E\big)$ for $\operatorname{Pol}\mathsf{U}$, we use the description of the pseudovariety $\operatorname{Pol}\mathsf{U}$ by pseudoinequalities from Proposition 2.2:

$$\operatorname{Pol}\mathsf{U} = [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid u, v \in \overline{\Omega}_A\mathsf{M} \text{ for some } A,\ \mathsf{U} \models u \leq v]\!]. \tag{4}$$

Let $t = f r^{\star} g$ be an arbitrary element of $E \cdot \{r, 1\} \cdot E$ with $f, g \in E$, $r^{\star} \in \{r, 1\}$. The set E is an idempotent of 2^M, hence, by Lemma 4.7, there exist elements $f_1, f_3, g_1, g_3 \in E$, $f_2, g_2 \in E \cap E(M)$ such that the equalities $f = f_1 f_2 f_3$, $g = g_1 g_2 g_3$ hold. By the ω-reducibility of (r, e, E) for $\big(\mathsf{U}^{\mathrm{d}}, \operatorname{Pol}\mathsf{U}\big)$, there exist ω-words $v_i \in \widehat{\alpha}^{-1}(f_i)$, $w_i \in \widehat{\alpha}^{-1}(g_i)$ satisfying $\operatorname{Pol}\mathsf{U} \models \overline{v} \leq v_i$, $\operatorname{Pol}\mathsf{U} \models \overline{v} \leq w_i$ for $i = 1, 2, 3$. Since we have $\mathsf{U} \subseteq \operatorname{Pol}\mathsf{U}$, the preceding properties imply that $\mathsf{U} \models \overline{v} \leq v_i$, $\mathsf{U} \models \overline{v} \leq w_i$ for $i = 1, 2, 3$. Furthermore, the relation $\mathsf{U}^{\mathrm{d}} \models \overline{u} \leq \overline{v}$ implies that $\mathsf{U} \models \overline{v} \leq \overline{u}$. Then we obtain relations $\mathsf{U} \models \overline{v}\,\overline{v}\,\overline{v} \leq v_3 \overline{u} w_1$ and $\mathsf{U} \models \overline{v}\,\overline{v}\,\overline{v} \leq v_2 v_3 w_1$, which imply that

$$\operatorname{Pol}\mathsf{U} \models (\overline{v}\,\overline{v}\,\overline{v})^{\omega+1} \leq (\overline{v}\,\overline{v}\,\overline{v})^{\omega}(v_3 \overline{u} w_1)(\overline{v}\,\overline{v}\,\overline{v})^{\omega}$$

and

$$\operatorname{Pol}\mathsf{U} \models (\overline{v}\,\overline{v}\,\overline{v})^{\omega+1} \leq (\overline{v}\,\overline{v}\,\overline{v})^{\omega}(v_2 v_3 w_1)(\overline{v}\,\overline{v}\,\overline{v})^{\omega},$$

respectively, by Relation (4). These relations are equivalent to the following:

$$\operatorname{Pol}\mathsf{U} \models (\overline{v})^{\omega+3} \leq (\overline{v})^{\omega}(v_3 \overline{u} w_1)(\overline{v})^{\omega}$$

and

$$\operatorname{Pol}\mathsf{U} \models (\overline{v})^{\omega+3} \leq (\overline{v})^{\omega}(v_2 v_3 w_1)(\overline{v})^{\omega},$$

respectively. These imply that

$$\begin{aligned}\operatorname{Pol}\mathsf{U} \models (\overline{v})^{\omega+5} = \overline{v}(\overline{v})^{\omega+3}\overline{v} &\leq v_1 (\overline{v})^{\omega}(v_3 \overline{u} w_1)(\overline{v})^{\omega} w_3 \leq \\ &\leq v_1 (v_2)^{\omega} v_3 \overline{u} w_1 (w_2)^{\omega} w_3\end{aligned} \tag{5}$$

and

$$\begin{aligned}\operatorname{Pol}\mathsf{U} \models (\overline{v})^{\omega+5} = \overline{v}(\overline{v})^{\omega+3}\overline{v} &\leq v_1 (\overline{v})^{\omega}(v_2 v_3 w_1)(\overline{v})^{\omega} w_3 \leq \\ &\leq v_1 (v_2)^{\omega+1} v_3 w_1 (w_2)^{\omega} w_3,\end{aligned} \tag{6}$$

respectively.

We put $u = (\overline{v})^{\omega+5}$, $v = \begin{cases} v_1 (v_2)^{\omega} v_3 \overline{u} w_1 (w_2)^{\omega} w_3 & \text{if } r^{\star} = r, \\ v_1 (v_2)^{\omega+1} v_3 w_1 (w_2)^{\omega} w_3 & \text{if } r^{\star} = 1. \end{cases}$

Then u, v are ω-words satisfying

- $\widehat{\alpha}(u) = e^{\omega+5} = e$,

- $$\widehat{\alpha}(v) = \begin{cases} f_1(f_2)^\omega f_3 r g_1 (g_2)^\omega g_3 = f_1 f_2 f_3 r g_1 g_2 g_3 = frg = fr^\star g = t & \text{if } r^\star = r, \\ f_1(f_2)^{\omega+1} f_3 g_1 (g_2)^\omega g_3 = f_1 f_2 f_3 g_1 g_2 g_3 = fg = fr^\star g = t & \text{if } r^\star = 1, \end{cases}$$

- $\mathsf{Pol}\,\mathsf{U} \models u \leq v$ by Properties (5) and (6),

with u independent of the choice of an element $t \in E \cdot \{r, 1\} \cdot E$, witnessing that the pair $\big(e, E \cdot \{r, 1\} \cdot E\big)$ is ω-reducible for Pol U, as required. □

To simplify the notation in proofs, we define a binary relation $\leq$ on the set $M \times 2^M$ in the following way:

$$(s, S) \leq (t, T) \quad \Leftrightarrow \quad s = t \text{ and } S \subseteq T. \tag{7}$$

It is obviously a partial order on $M \times 2^M$.

The following corollary is inspired by a similar property formulated in terms of *covering* of regular languages, which was proven in [25, Proof of Lemma 8.13].

Corollary 4.13 *Let* U *be an arbitrary pseudovariety. Let* $R \in 2^M$, $e \in E(M)$, $E \in E(2^M)$. *If, for every element* $r \in R$, *the triple* (r, e, E) *is* ω*-reducible for* $(\mathsf{U}^{\mathrm{d}}, \mathsf{Pol}\,\mathsf{U})$, *then the pair* (e, ERE) *is* ω*-reducible for* Pol U.

Proof By Lemma 4.12, for every $r \in R$, the pair $\big(e, E \cdot \{r, 1\} \cdot E\big)$ is ω-reducible for Pol U. Then we have

$$(e, ERE) \leq \Big(e, \prod_{r \in R} (E \cdot \{r, 1\} \cdot E)\Big) = \prod_{r \in R} \big(e, E \cdot \{r, 1\} \cdot E\big).$$

Hence, the pair (e, ERE) is ω-reducible for Pol U by Lemmas 4.9 and 4.10. □

Now we get back to our fixed *locally finite* pseudovariety V. Recall that we have fixed also a surjective homomorphism $\alpha\colon A^* \to M$ into a finite monoid M. For the following corollary, we need to assume additionally that the homomorphism α is V-*compatible* (for the definition of V-compatibility, see Subsection 3.1.1). Then, for every $s \in M$, we denote the equivalence class $[\widehat{\alpha}^{-1}(s)]_{\sim_{\mathsf{V}}^{A}}$ simply by $[s]_{\sim_{\mathsf{V}}^{A}}$.

The following corollary was also proven in [37, Theorem 4.7] for the case when the pseudovariety V is selfdual. In this paper, we prove it easily using Corollary 4.13.

Corollary 4.14 *Let* V *be a locally finite pseudovariety. Let* $\alpha\colon A^* \to M$ *be a surjective* V*-compatible homomorphism into a finite monoid* M. *Let* $(e, E) \in E(M \times 2^M)$ *be an* ω*-reducible pair for* Pol V. *Let*

$$S_e = \big\{s \in M \mid [e]_{\sim_{\mathsf{V}}^{A}} \leq [s]_{\sim_{\mathsf{V}}^{A}}\big\}.$$

Then the pair $\big(e, E \cdot S_e \cdot E\big)$ *is also* ω*-reducible for* Pol V.

Proof Let $(e, E) \in E(M \times 2^M)$ be an ω-reducible pair for $\mathrm{Pol}\,\mathsf{V}$. Then there exist an ω-word $x \in \Omega_A^\omega \mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^\omega \mathsf{M}$ such that

i) $\mathrm{Pol}\,\mathsf{V} \models x \le X$,
ii) $\widehat{\alpha}(x) = e, \widehat{\alpha}(X) = E$.

Let $s \in S_e$ be an arbitrary element. Choose a word $z \in \widehat{\alpha}^{-1}(s)$. Then, by the definition of the set S_e, we have $\mathsf{V} \models x \le z$, i.e., $\mathsf{V}^{\mathrm{d}} \models z \le x$. Using Conditions i) and ii), we obtain that the triple (s, e, E) is ω-reducible for $(\mathsf{V}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{V})$. By Corollary 4.13, we obtain that the pair $\big(e, E \cdot S_e \cdot E\big)$ is ω-reducible for $\mathrm{Pol}\,\mathsf{V}$, as required. ◻

4.3.2 Employment of Lemma 4.5

Now we get to the announced application of Lemma 4.5. Since we will use Corollary 4.14 in the proof, we need to assume again that the homomorphism $\alpha\colon A^* \to M$ is V-compatible.

Lemma 4.15 *Let* V *be a locally finite pseudovariety. Let* $\alpha\colon A^* \to M$ *be a surjective* V*-compatible homomorphism onto a finite monoid* M*. Let* $n, k \in \mathbb{N}$, $k \ge 2^{|M|}$*. Let* $u_1, \ldots, u_n \in A^*$ *be words such that* $\alpha(u_1) = \alpha(u_2) = \cdots = \alpha(u_n) = e$ *for some idempotent* $e \in E(M)$*. Let* $U, U_1, \ldots, U_n \subseteq A^*$ *be sets satisfying the assumptions of Lemma 4.5. Let* $W_1, \ldots, W_m \subseteq A^*$*, where* $m \in \mathbb{N}$*, be sets satisfying Conditions 1 and 2 of Lemma 4.5. Let* U *be an arbitrary pseudovariety. If the pairs*

$$\big(e, \alpha(U_1)\big), \ldots, \big(e, \alpha(U_n)\big)$$

are ω*-reducible for* $\mathrm{Pol}\,\mathsf{U}$ *and at least one of the following two conditions is satisfied:*

1. $\mathsf{U} = \mathsf{V}$ *or*
2. *for every set* W_i *of the form (ii) and for every word* $\widetilde{v} \in \overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}$*, the triple* $\big(\alpha(\widetilde{v}), e, E\big)$ *is* ω*-reducible for* $(\mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U})$*,*

then the pair $\big(e, \alpha(U)\big)$ *is also* ω*-reducible for* $\mathrm{Pol}\,\mathsf{U}$.

Proof By Condition 1 of Lemma 4.5, we know that $U \subseteq W_1 \cdots W_m$. This implies that

$$\big(e, \alpha(U)\big) \le \big(e, \alpha(W_1) \cdots \alpha(W_m)\big) = \big(e, \alpha(W_1)\big) \cdots \big(e, \alpha(W_m)\big).$$

Hence, by Lemmas 4.9 and 4.10, it suffices to show that, for every $i \in \{1, \ldots, m\}$, the pair $\big(e, \alpha(W_i)\big)$ is ω-reducible for $\mathrm{Pol}\,\mathsf{U}$.

Let $i \in \{1, \ldots, m\}$ be arbitrary. If the set W_i is of the form (i), we have

$$\big(e, \alpha(W_i)\big) \le \big(e, \alpha(U_\iota)\big) \cdots \big(e, \alpha(U_\kappa)\big).$$

Since the pairs $\big(e, \alpha(U_\iota)\big), \ldots, \big(e, \alpha(U_\kappa)\big)$ are ω-reducible for $\mathrm{Pol}\,\mathsf{U}$ by the assumption, we obtain that the pair $\big(e, \alpha(W_i)\big)$ is also ω-reducible for $\mathrm{Pol}\,\mathsf{U}$, using Lemmas 4.9 and 4.10.

Now suppose that the set W_i is of the form (ii). At first, since we have

$$(e, E) = \Big(e, \alpha\big((U_{\kappa+1}\cdots U_\lambda)^K\big)\Big) = \Big(e, \alpha(U_{\kappa+1}\cdots U_\lambda)\Big)^K = \\ = \Big(\big(e, \alpha(U_{\kappa+1})\big)\cdots\big(e, \alpha(U_\lambda)\big)\Big)^K \tag{8}$$

and the pairs $\big(e, \alpha(U_{\kappa+1})\big), \ldots, \big(e, \alpha(U_\lambda)\big)$ are ω-reducible for Pol U by the assumption, we obtain that the pair (e, E) is also ω-reducible for Pol U, using Lemma 4.10. We distinguish the two cases:

1. $\mathsf{U} = \mathsf{V}$,
2. for every word $\widetilde{v} \in \overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}$, the triple $\big(\alpha(\widetilde{v}), e, E\big)$ is ω-reducible for $(\mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U})$.

In the first case when $\mathsf{U} = \mathsf{V}$, we have $\mathrm{Pol}_{k-2^{|M|}}\,\mathsf{U} \models u_{\kappa+1}\ldots u_{\iota'-1} \le \overline{U}$ by Lemma 4.5. Further, recall the relations

$$\mathrm{Pol}_{k-2^{|M|}}\,\mathsf{U} \models u_i \le U_i, \quad i = 1, \ldots, n$$

from the assumptions of Lemma 4.5. Altogether, we obtain the relation

$$\mathrm{Pol}_{k-2^{|M|}}\,\mathsf{U} \models u_{\kappa+1}\ldots u_{\kappa'} = (u_{\kappa+1}\ldots u_{\iota'-1}) \cdot u_{\iota'}\ldots u_{\kappa'} \le \overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}.$$

This implies that

$$\mathsf{U} \models u_{\kappa+1}\ldots u_{\kappa'} \le \overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}.$$

Since $\alpha(u_{\kappa+1}\ldots u_{\kappa'}) = e$, we obtain that $[e]_{\sim^A_{\mathsf{U}}} \le [\widetilde{v}]_{\sim^A_{\mathsf{U}}}$ for every word $\widetilde{v} \in \overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}$. This means that

$$\alpha(\overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}) \subseteq S_e := \big\{s \in M \mid [e]_{\sim^A_{\mathsf{U}}} \le [s]_{\sim^A_{\mathsf{U}}}\big\}. \tag{9}$$

Then, by Corollary 4.14 and Lemma 4.9, the pair

$$\big(e, E \cdot \alpha(\overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}) \cdot E\big) \le (e, E S_e E)$$

is ω-reducible for Pol U.

In the second case, for every word $\widetilde{v} \in \overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}$, the triple $\big(\alpha(\widetilde{v}), e, E\big)$ is ω-reducible for $(\mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U})$ by the assumption. Then, by Corollary 4.13, we obtain that the pair $\big(e, E \cdot \alpha(\overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}) \cdot E\big)$ is ω-reducible for Pol U.

Finally, in both cases, we have

$$\alpha(W_i) = \alpha(U_\iota \cdots U_\kappa) \cdot \alpha(\overline{U}) \cdot \alpha(U_{\iota'} \cdots U_{\kappa'}) = \\ = \alpha(U_\iota \cdots U_\kappa) \cdot E \cdot \alpha(\overline{U}) \cdot \alpha(U_{\iota'} \cdots U_{\kappa'}) \cdot E = \\ = \alpha(U_\iota) \cdots \alpha(U_\kappa) \cdot E \cdot \alpha(\overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}) \cdot E.$$

Then the pair

$$\big(e, \alpha(W_i)\big) = \big(e, \alpha(U_\iota)\big) \cdots \big(e, \alpha(U_\kappa)\big) \cdot \big(e, E \cdot \alpha(\overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}) \cdot E\big)$$

is ω-reducible for Pol U by Lemma 4.10. We have finished the proof of Lemma 4.15. □

Remark 4.16

1. In the case when $\mathsf{U} = \mathsf{V}$, one can prove that Condition 2 of Lemma 4.15 is also satisfied. The procedure is similar to the proof of Corollary 4.14. It suffices to use
   - Relation (9), which implies that, for every word $\widetilde{v} \in \overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}$ and every ω-word $x \in \widehat{\alpha}^{-1}(e)$, we have $\mathsf{U}^{\mathrm{d}} \models \widetilde{v} \leq x$,
   - the ω-reducibility of the pair (e, E), following from Relation (8), and
   - Corollary 4.13.
2. In the case when $\mathsf{U} \neq \mathsf{V}$, the assumption of V-compatibility of α is superfluous.

4.4 Main theorem

Now we proceed to prove the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$. The procedure of the proof is the same as in the author's PhD thesis [36, Subsection 3.2.2]. However, the presentation differs in part due to the employment of Lemma 4.15 in this paper.

In the proof, we use the stratification of the pseudovariety Pol V from Subsection 3.3.2. We choose an appropriate index ν_0 for which

$$\mathcal{P}_{\mathrm{Pol}_{\nu_0}\mathsf{V}}[M] = \mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M] \tag{10}$$

(such an index must exist by Proposition 3.7). The whole proof is done using the locally finite pseudovarieties $\mathrm{Pol}_\nu \mathsf{V}$ for $\nu \leq \nu_0$. However, we do not see that Relation (10) holds until finishing the proof.

Theorem 4.17 *Let* V *be a locally finite pseudovariety of ordered monoids and let* M *be a finite monoid. Then the set* $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ *is* ω*-reducible.*

Moreover, if $\nu_0 = 1 + 9|M| \cdot 2^{|M|}$ *and there is a surjective* V*-compatible homomorphism* $\alpha\colon A^* \to M$*, then the equality* $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M] = \mathcal{P}_{\mathrm{Pol}_{\nu_0}\mathsf{V}}[M]$ *holds.*

Proof By Lemma 3.1, to show that the set $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ is ω-reducible, it suffices to consider the case when we have a surjective V-compatible homomorphism $\alpha\colon A^* \to M$.

To prove the whole theorem, we use Proposition 3.4. By Proposition 3.11, $\mathrm{Pol}_{\nu_0}\mathsf{V}$ is a locally finite pseudovariety such that $\mathrm{Pol}_{\nu_0}\mathsf{V} \subseteq \mathrm{Pol}\,\mathsf{V}$. We show that the assumptions of Proposition 3.4 for the locally finite pseudovariety $\mathrm{Pol}_{\nu_0}\mathsf{V}$, seen as a subpseudovariety of Pol V, are satisfied. This will imply, by Proposition 3.4, precisely that the statement of Theorem 4.17 holds.

Let $u \in A^*$ be an arbitrary word and $U \subseteq A^*$ be an arbitrary set of words satisfying the relation $\mathrm{Pol}_{\nu_0}\mathsf{V} \models u \leq U$. We need to show that there exist an ω-word $x \in \Omega_A^\omega \mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^\omega \mathsf{M}$ satisfying the following conditions:

i) $\mathrm{Pol}\,\mathsf{V} \models x \leq X$,
ii) $\widehat{\alpha}(x) = \alpha(u), \widehat{\alpha}(X) = \alpha(U)$.

We prove this by the induction on the height of the word u in a fixed factorization forest d for α. More precisely, we prove the following theorem.

Let d be a factorization forest for α. For every $u \in A^*$, denote

$$\nu(u) := 1 + 3 \cdot 2^{|M|} \cdot h_d(u).$$

Theorem 4.18 *Let d be a factorization forest for α. Let $u \in A^*$, $U \subseteq A^*$. Let* $\mathrm{Pol}_{\nu(u)}\,\mathsf{V} \models u \leq U$. *Then the pair $\big(\alpha(u), \alpha(U)\big)$ is ω-reducible for* $\mathrm{Pol}\,\mathsf{V}$.

At first, let us show how to use Theorem 4.18 to complete the proof of Theorem 4.17. Choose a factorization forest d for α of height at most $3|M|$, which exists by Theorem 2.4. Then

$$\nu(u) = 1 + 3 \cdot 2^{|M|} \cdot h_d(u) \leq 1 + 9|M| \cdot 2^{|M|} = \nu_0.$$

This implies that the relation $\mathrm{Pol}_{\nu(u)}\,\mathsf{V} \models u \leq U$ holds. By Theorem 4.18, we obtain that the pair $\big(\alpha(u), \alpha(U)\big)$ is ω-reducible for $\mathrm{Pol}\,\mathsf{V}$. This means that there exist an ω-word $x \in \Omega_A^{\omega}\mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^{\omega}\mathsf{M}$ satisfying Conditions i) and ii). We have proven that the assumptions of Proposition 3.4 are satisfied. Hence, by Proposition 3.4, we obtain that $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M] = \mathcal{P}_{\mathrm{Pol}_{\nu_0}\mathsf{V}}[M]$ and the set $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ is ω-reducible. $\square$

It remains to prove Theorem 4.18. We use Lemmas 4.9, 4.10, and 4.15 to show that the pair $\big(\alpha(u), \alpha(U)\big)$ is ω-reducible for $\mathrm{Pol}\,\mathsf{V}$ for every pair $(u, U) \in A^* \times 2^{A^*}$ satisfying the assumptions of Theorem 4.18.

We start with the case when $h_d(u) = 0$, i.e., $u \in A \cup \{\varepsilon\}$ and $\nu(u) = 1$. This is solved in the following lemma.

Lemma 4.19 *Let $u \in A \cup \{\varepsilon\}$ and $U \subseteq A^*$ be such that* $\mathrm{Pol}_1\,\mathsf{V} \models u \leq U$. *Then* $\mathrm{Pol}\,\mathsf{V} \models u \leq U$ *and the pair $\big(\alpha(u), \alpha(U)\big)$ is ω-reducible for* $\mathrm{Pol}\,\mathsf{V}$.

Proof We prove the relation $\mathrm{Pol}\,\mathsf{V} \models u \leq U$. This will imply directly the ω-reducibility (even the word reducibility) of $\big(\alpha(u), \alpha(U)\big)$ for $\mathrm{Pol}\,\mathsf{V}$. We divide the proof into two parts depending whether $u = \varepsilon$ or $u = a \in A$.

1. $u = \varepsilon$
   Let $L \in Pol(\mathcal{V})(A)$ be an arbitrary language such that $\varepsilon \in L$. By the definition of $Pol(\mathcal{V})(A)$, there exists a language $L' \in \mathcal{V}(A)$ such that $\varepsilon \in L' \subseteq L$. The relation $\mathrm{Pol}_1\,\mathsf{V} \models \varepsilon \leq U$ implies that $\mathsf{V} \models \varepsilon \leq U$. Hence we obtain that $U \subseteq L' \subseteq L$. We have proven the desired relation $\mathrm{Pol}\,\mathsf{V} \models \varepsilon \leq U$.
2. $u = a \in A$
   We proceed analogously to the previous case. Let $L \in Pol(\mathcal{V})(A)$ be an arbitrary language such that $a \in L$. By the definition of $Pol(\mathcal{V})(A)$, there exists a language $L' \in Pol_1(\mathcal{V})(A)$ such that $a \in L' \subseteq L$. Since $\mathrm{Pol}_1\,\mathsf{V} \models a \leq U$, we obtain that $U \subseteq L' \subseteq L$. We have proven the desired relation $\mathrm{Pol}\,\mathsf{V} \models a \leq U$.

□

Now we turn back to the proof of Theorem 4.18.

Proof of Theorem 4.18 The proof goes by the induction on the height of u in the factorization forest d. If $h_d(u) = 0$, then $u \in A \cup \{\varepsilon\}$. By the assumption, we have $\mathrm{Pol}_1 \mathsf{V} \models u \leq U$. By Lemma 4.19, we obtain that the pair $\big(\alpha(u), \alpha(U)\big)$ is ω-reducible for $\mathrm{Pol}\,\mathsf{V}$.

Suppose that $h_d(u) \geq 1$. Let $d(u) = (u_1, \ldots, u_n)$. Then two cases can occur:

a) $n = 2$,
b) $n > 2$.

The proceeding in Case a) is straightforward. Let $d(u) = (u_1, u_2)$. For $i = 1, 2$, denote by U_i the maximal subset of A^* satisfying $\mathrm{Pol}_{\nu(u_i)} \mathsf{V} \models u_i \leq U_i$. Then, by the induction assumption, the pairs $\big(\alpha(u_1), \alpha(U_1)\big)$, $\big(\alpha(u_2), \alpha(U_2)\big)$ are ω-reducible for $\mathrm{Pol}\,\mathsf{V}$. We have

$$\begin{aligned}\nu(u_i) &= 1 + 3 \cdot 2^{|M|} \cdot h_d(u_i) \leq 1 + 3 \cdot 2^{|M|} \cdot (h_d(u) - 1) = \\ &= 1 + 3 \cdot 2^{|M|} \cdot h_d(u) - 3 \cdot 2^{|M|} = \nu(u) - 3 \cdot 2^{|M|} < \nu(u) - 1.\end{aligned}$$

Then, by Lemma 4.4, we obtain that $U \subseteq U_1 U_2$. This implies that

$$\big(\alpha(u), \alpha(U)\big) \leq \big(\alpha(u_1), \alpha(U_1)\big) \cdot \big(\alpha(u_2), \alpha(U_2)\big).$$

Using Lemmas 4.9 and 4.10, we obtain that the pair $\big(\alpha(u), \alpha(U)\big)$ is ω-reducible for $\mathrm{Pol}\,\mathsf{V}$.

Now suppose that $h_d(u) \geq 1$ and $d(u) = (u_1, \ldots, u_n)$, where $n > 2$. Then there exists an idempotent $e \in E(M)$ such that $\alpha(u_1) = \cdots = \alpha(u_n) = \alpha(u) = e$. To complete the proof in this case, we use Lemma 4.15. Let

$$k := \nu(u) - 2 \cdot 2^{|M|}.$$

Then we have

$$\begin{aligned}k &= 1 + 3 \cdot 2^{|M|} \cdot h_d(u) - 2 \cdot 2^{|M|} \geq \\ &\geq 1 + 3 \cdot 2^{|M|} - 2 \cdot 2^{|M|} = \\ &= 1 + 2^{|M|} > 2^{|M|}\end{aligned}$$

and

$$\begin{aligned}k + 2 \cdot \mathrm{i}_{k-2^{|M|}}(u_1, \ldots, u_n) &= \nu(u) - 2 \cdot 2^{|M|} + 2 \cdot \mathrm{i}_{k-2^{|M|}}(u_1, \ldots, u_n) \leq \\ &\leq \nu(u) - 2 \cdot 2^{|M|} + 2 \cdot 2^{|M|} = \\ &= \nu(u).\end{aligned}$$

By the assumption of Theorem 4.18, we have $\mathrm{Pol}_{\nu(u)}\,\mathsf{V} \models u \le U$. By the preceding, this implies that $\mathrm{Pol}_{k+2\cdot \mathrm{i}_{k-2^{|M|}}(u_1,\dots,u_n)}\,\mathsf{V} \models u \le U$.

Further, for $i = 1, \dots, n$, let U_i be the maximal subset of A^* satisfying

$$\mathrm{Pol}_{k-2^{|M|}}\,\mathsf{V} \models u_i \le U_i.$$

Since

$$k - 2^{|M|} = \nu(u) - 2 \cdot 2^{|M|} - 2^{|M|} = \nu(u) - 3 \cdot 2^{|M|}$$

and

$$\begin{aligned}\nu(u_i) &= 1 + 3 \cdot 2^{|M|} \cdot h_d(u_i) \le \\ &\le 1 + 3 \cdot 2^{|M|} \cdot (h_d(u) - 1) = 1 + 3 \cdot 2^{|M|} \cdot h_d(u) - 3 \cdot 2^{|M|} = \\ &= \nu(u) - 3 \cdot 2^{|M|},\end{aligned}$$

we obtain the relations $\mathrm{Pol}_{\nu(u_i)}\,\mathsf{V} \models u_i \le U_i$ for $i = 1, \dots, n$. Then, by the induction assumption, the pairs

$$\big(\alpha(u_1), \alpha(U_1)\big), \dots, \big(\alpha(u_n), \alpha(U_n)\big)$$

are ω-reducible for $\mathrm{Pol}\,\mathsf{V}$. By Lemma 4.15, we obtain that the pair $\big(\alpha(u_1 \dots u_n), \alpha(U)\big) = \big(\alpha(u), \alpha(U)\big)$ is also ω-reducible for $\mathrm{Pol}\,\mathsf{V}$, as required. □

The following corollary was proven already in the author's paper [37, Theorem 5.1].

Corollary 4.20 ([37]). *For every concatenation hierarchy with a locally finite basic pseudovariety* V_0*, the pseudovariety* $\mathsf{V}_{1/2}$ *corresponding to level* $1/2$ *is* ω*-reducible.*

Proof It follows from Theorem 4.17, from the definition of the ω-reducibility of pseudovarieties, and from the relation $\mathsf{V}_{1/2} = \mathrm{Pol}\,\mathsf{V}_0$. □

Let U be an arbitrary pseudovariety. We remind of the characterization of the pseudovariety $\mathrm{Pol}\,\mathsf{U}$ by pseudoinequalities from Proposition 2.2:

$$\mathrm{Pol}\,\mathsf{U} = [\![u^{\omega+1} \le u^\omega v u^\omega \mid u, v \in \overline{\Omega}_A\mathsf{M} \text{ for some } A,\ \mathsf{U} \models u \le v]\!].$$

If we choose $\mathsf{U} = \mathrm{Pol}\,\mathsf{V}$, we obtain

$$\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}} = [\![u^{\omega+1} \le u^\omega v u^\omega \mid u, v \in \overline{\Omega}_A\mathsf{M} \text{ for some } A,\ \mathrm{Pol}\,\mathsf{V} \models v \le u]\!]. \quad (11)$$

The following corollary was proven already in the author's paper [37, Corollary 5.2].

Corollary 4.21 ([37]). *For a concatenation hierarchy with a locally finite basic pseudovariety* V_0*, there is the following basis of* ω*-inequalities for the pseudovariety* $\mathsf{V}_{3/2}$*:*

$$\mathsf{V}_{3/2} = [\![u^{\omega+1} \le u^\omega v u^\omega \mid u, v \in \Omega^\omega_A\mathsf{M} \textit{ for some } A,\ \mathsf{V}_{1/2} \models v \le u]\!].$$

Proof It follows from the relations $\mathsf{V}_{1/2} = \mathrm{Pol}\,\mathsf{V}_0$, $\mathsf{V}_{3/2} = \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V}_0)^{\mathrm{d}}$, Relation (11), and Corollary 4.20. □

4.5 Algorithm computing $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$

In [25], there is an algorithm computing the set $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$, where V is a locally finite selfdual pseudovariety. We explain a connection between this algorithm and our proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$.

The following definition of a set $\mathcal{A}_{\mathrm{Pol}\,\mathsf{V}}[M]$ is adopted from [25, Subsection 6.2], where it is described in terms of *covering* of regular languages. In our form, it can be found in the author's paper [37, Subsection 4.2].

Let $\mathcal{A}_{\mathrm{Pol}\,\mathsf{V}}[M]$ be the smallest set $\mathcal{A} \subseteq M \times 2^M$ satisfying the following conditions:

1. $\forall s \in M\colon \big(s, \{s\}\big) \in \mathcal{A}$,
2. $\forall (s, S) \in \mathcal{A}, \forall S' \subseteq S\colon (s, S') \in \mathcal{A}$,
3. $\forall (s_1, S_1), (s_2, S_2) \in \mathcal{A}\colon (s_1 s_2, S_1 S_2) \in \mathcal{A}$,
4. $\forall (e, E) \in \mathcal{A} \cap E(M \times 2^M)\colon \big(e, E \cdot S_e \cdot E\big) \in \mathcal{A}$, where

$$S_e = \Big\{s \in M \mid [s]_{\sim^A_{\mathsf{V}_0}} = [e]_{\sim^A_{\mathsf{V}_0}}\Big\}.$$

Theorem 4.22 *The equality* $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M] = \mathcal{A}_{\mathrm{Pol}\,\mathsf{V}}[M]$ *holds.*

Proof The equality follows from [25, Theorem 6.5]. More precisely, this description of the set $\mathcal{A}_{\mathrm{Pol}\,\mathsf{V}}[M]$ is a translation of the original algorithm from [25], where it is described in terms of *covering* of regular languages, into the terms of generalized $(\mathrm{Pol}\,\mathsf{V})$-pointlike sets. For more details, see [37, p. 115, Footnote ^{l}]. □

An alternative proof of Theorem 4.22, which does not use results of [25], can be obtained by results of this section. More precisely, the inclusion $\mathcal{A}_{\mathrm{Pol}\,\mathsf{V}}[M] \subseteq \mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ follows from the ω-reducibility for $\mathrm{Pol}\,\mathsf{V}$ of the set $\mathcal{A}_{\mathrm{Pol}\,\mathsf{V}}[M]$, which was proven by Lemmas 4.8, 4.9, 4.10 and Corollary 4.14. The inclusion $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M] \subseteq \mathcal{A}_{\mathrm{Pol}\,\mathsf{V}}[M]$ follows from the fact that every pair $(s, S) \in \mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ can be built by Rules 1–4 of the algorithm computing $\mathcal{A}_{\mathrm{Pol}\,\mathsf{V}}[M]$ above – this fact is implicitly included in the proof of Theorem 4.17. Indeed, in this proof, we showed by induction that every "non-simple" pair $(s, S) \in \mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ can be built by some "simpler" pairs ω-reducible for $\mathrm{Pol}\,\mathsf{V}$ and then the ω-reducibility of (s, S) for $\mathrm{Pol}\,\mathsf{V}$ was derived using Lemmas 4.9, 4.10, and 4.15. Furthermore, Lemma 4.15 was proven using Lemmas 4.9, 4.10 and Corollary 4.14, which show that every pair built from pairs ω-reducible for $\mathrm{Pol}\,\mathsf{V}$ by Rules 2–4 of the algorithm is also ω-reducible for $\mathrm{Pol}\,\mathsf{V}$.

5 ω-reducibility of $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$

In this section, we prove the ω-reducibility of sets of the form $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$, where the pseudovariety V is locally finite. The procedure of the proof will be analogous as for sets of the form $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ in Section 4, although with a more complex structure.

Let V be a locally finite pseudovariety. To prove the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$, we use the stratification of $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ from Subsection 3.3.3 and Proposition 3.4 for the locally finite pseudovariety $\mathsf{W} = \mathrm{Pol}_l(\mathrm{Pol}_k\,\mathsf{V})^{\mathrm{d}}$, where k, l are specific numbers such that $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M] = \mathcal{P}_{\mathrm{Pol}_l(\mathrm{Pol}_k\,\mathsf{V})^{\mathrm{d}}}[M]$.

In the first subsection, we define *k-factorial pairs for* U, where k is an arbitrary nonnegative integer and U is an arbitrary pseudovariety, and we show their basic properties. The employment of k-factorial pairs in our proof of the ω-reducibility simplifies the presentation.

In the second subsection, we present a special multiplication on a set of the form $2^{M\times 2^M}$, where M is an arbitrary monoid. We use this multiplication in the following subsections.

In the third subsection, we define a (k,l)*-index* of a sequence of words and state its basic properties.

In the fourth subsection, we prove an important lemma, which describes a special way of a simultaneous “factorization” of an inequality of the form $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \ldots u_n \leq v$ and an l-factorial pair (v, V) for X, where n is arbitrarily large, $u_1, \ldots, u_n, v \in A^*$, $V \subseteq A^*$, and X is an arbitrary pseudovariety. This lemma is crucial for our proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ by a factorization forest.

Then in the fifth subsection, we show how to prove the ω-reducibility for $(\mathrm{Pol}\,\mathsf{V}, \mathsf{X})$ of a pair $\big(\alpha(u_1 \ldots u_n), \alpha(\mathcal{U})\big) \in M \times 2^{M\times 2^M}$, where $\alpha\colon A^* \to M$ is a surjective homomorphism, $\alpha(u_1) = \cdots = \alpha(u_n) = e \in E(M)$, and for every pair $(v, V) \in \mathcal{U}$, the relation $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \ldots u_n \leq v$ holds and the pair (v, V) is l-factorial for X. The proof uses the ω-reducibility of special “factors” of the pair $\big(e, \alpha(\mathcal{U})\big) = \big(\alpha(u_1 \ldots u_n), \alpha(\mathcal{U})\big)$, created by the lemma mentioned above. Moreover, by addition of an extra condition, we are able to extend this process to the ω-reducibility for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$, where a pseudovariety U is not required to be locally finite.

Finally, in the sixth subsection, we perform the proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$, using the results of the fifth subsection and the induction on the height of words in a factorization forest.

In the last (seventh) subsection, we describe a connection between our proof of the ω-reducibility and the algorithm computing the set $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ from [25].

5.1 *k*-factorial pairs

For this subsection, fix an alphabet A, a finite monoid M, and a surjective homomorphism $\alpha\colon A^* \to M$.

Definition 5.1 Let U be a pseudovariety of ordered monoids.

- A pair $(u, U) \in A^* \times 2^{A^*}$ is 0*-factorial for* U if the pair $\big(\alpha(u), \alpha(U)\big)$ is ω-reducible for U.
- Let $k \in \mathbb{N}$. A pair $(u, U) \in A^* \times 2^{A^*}$ is *k-factorial for* U if, for every factorization $u = u'u''$, where $u', u'' \in A^*$, there exist sets of words $U', U'' \in 2^{A^*}$ such that
  - $U \subseteq U'U''$,
  - the pairs (u', U'), (u'', U'') are $(k-1)$-factorial for U.

The definition of k-factorial pairs describes precisely a property of pairs $(u, U) \in A^* \times 2^{A^*}$ that we will need at some point in our proof of the ω-reducibility of $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. We will use k-factorial pairs (u, U) for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ in connection with relations of the form $\mathrm{Pol}_l\,\mathsf{W} \models u \leq U$ for a specific locally finite pseudovariety $\mathsf{W} \subseteq \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ and a specific number l related to k.

Generally, if $\mathsf{U} = \mathrm{Pol}\,\mathsf{V}$ for a locally finite pseudovariety V, k-factorial pairs (u, U) for U have analogous properties as inequalities of the form $\mathrm{Pol}_k\,\mathsf{V} \models u \leq U$ have – as we can see in the following lemmas.

Recall the partial order $\leq$ on $M \times 2^M$ defined in (7):

$$(s, S) \leq (t, T) \quad \Leftrightarrow \quad s = t \;\text{ and }\; S \subseteq T. \tag{12}$$

We will use this notation, defined in the same way, also for $(s, S), (t, T) \in A^* \times 2^{A^*}$.

Lemma 5.2 *Let $k \in \mathbb{N}_0$ be an arbitrary number. Then the following three statements hold:*

1. *Let $(u, U) \in A^* \times 2^{A^*}$ be a k-factorial pair for* U. *Let $V \subseteq U$ be an arbitrary subset. Then the pair (u, V) is also k-factorial for* U.
2. *Let $(u_1, U_1), (u_2, U_2) \in A^* \times 2^{A^*}$ be k-factorial pairs for* U. *Then the pair (u_1u_2, U_1U_2) is also k-factorial for* U.
3. *Let $(u, U) \in A^* \times 2^{A^*}$ be a $(k+1)$-factorial pair for* U. *Then (u, U) is also k-factorial for* U.

Proof We prove all the three statements simultaneously by induction on k.

At first, let $k = 0$.

1. Let (u, U) be a 0-factorial pair for U. By the definition, this means that the pair $\big(\alpha(u), \alpha(U)\big)$ is ω-reducible for U. Then, for a subset $V \subseteq U$, the pair $\big(\alpha(u), \alpha(V)\big)$ is also ω-reducible for U by Lemma 4.9. This means that (u, V) is 0-factorial for U.
2. Let $(u_1, U_1), (u_2, U_2)$ be 0-factorial pairs for U. This means that the pairs $\big(\alpha(u_1), \alpha(U_1)\big)$, $\big(\alpha(u_2), \alpha(U_2)\big)$ are ω-reducible for U. Then the pair $\big(\alpha(u_1u_2), \alpha(U_1U_2)\big)$ is also ω-reducible for U by Lemma 4.10. This means that the pair (u_1u_2, U_1U_2) is 0-factorial for U.
3. Let (u, U) be an 1-factorial pair for U. Let $u', u'' \in A^*$ be some words such that $u = u'u''$. Then, by the definition, there exist sets of words $U', U'' \in 2^{A^*}$ such that
   - $U \subseteq U'U''$,
   - the pairs $(u', U'), (u'', U'')$ are 0-factorial for U, i.e., the pairs $\big(\alpha(u'), \alpha(U')\big)$, $\big(\alpha(u''), \alpha(U'')\big)$ are ω-reducible for U.

   Then the pair

   $$\big(\alpha(u), \alpha(U)\big) \leq \big(\alpha(u') \cdot \alpha(u''), \alpha(U') \cdot \alpha(U'')\big)$$

   is ω-reducible for U by Lemmas 4.9 and 4.10. This means that the pair (u, U) is 0-factorial for U.

Now suppose that $k > 0$.

1. Let (u, U) be a k-factorial pair for U and $V \subseteq U$ be an arbitrary subset. Then, by the definition, for every factorization $u = u'u''$, where $u', u'' \in A^*$, there exist sets of words $U', U'' \in 2^{A^*}$ such that
   - $U \subseteq U'U''$,
   - the pairs (u', U'), (u'', U'') are $(k-1)$-factorial for U.

   Then also $V \subseteq U \subseteq U'U''$. This implies that the pair (u, V) is also k-factorial for U.
2. Let (u_1, U_1), (u_2, U_2) be k-factorial pairs for U. Let $u', u'' \in A^*$ be arbitrary words such that $u_1u_2 = u'u''$. To prove that the pair (u_1u_2, U_1U_2) is k-factorial U, we need to find sets of words $U', U'' \in 2^{A^*}$ such that
   - $U_1U_2 \subseteq U'U''$,
   - the pairs (u', U'), (u'', U'') are $(k-1)$-factorial for U.

   We can assume without loss of generality that u' is a prefix of u_1. The second case, when u'' is a suffix of u_2, is symmetrical. When u' is a prefix of u_1, there exists a word $u_1'' \in A^*$ such that $u_1 = u'u_1''$ and $u'' = u_1''u_2$. Since the pair (u_1, U_1) is k-factorial for U by the assumption, there exist sets of words $U', U_1'' \in 2^{A^*}$ such that
   - $U_1 \subseteq U'U_1''$,
   - the pairs (u', U'), (u_1'', U_1'') are $(k-1)$-factorial for U.

   Further, since the pair (u_2, U_2) is k-factorial for U by the assumption, it is also $(k-1)$-factorial for U, using the induction assumption. Define $U'' := U_1''U_2$. Then we obtain that
   - $U_1U_2 \subseteq U'U_1''U_2 = U'U''$,
   - the pairs (u', U'), (u_1'', U_1''), and (u_2, U_2) are $(k-1)$-factorial for U.

   Finally, by the induction assumption, the pair $(u'', U'') = (u_1''u_2, U_1''U_2)$ is also $(k-1)$-factorial for U.
3. Let (u, U) be a $(k+1)$-factorial pair U. Let $u', u'' \in A^*$ be arbitrary words such that $u = u'u''$. Then, by the definition, there exist sets of words $U', U'' \in 2^{A^*}$ such that
   - $U \subseteq U'U''$,
   - the pairs (u', U'), (u'', U'') are k-factorial for U.

   Then, by Items 1 and 2, the pair $(u, U) \leq (u'u'', U'U'')$ is also k-factorial for U.

□

The following lemma is analogous to Lemma 3.13 on the factorization of inequalities of the form $\mathrm{Pol}_k \mathsf{V} \models u_1 \dots u_n \leq v$, where $u_1, \dots, u_n, v \in A^*$ (more precisely, to a generalization of Lemma 3.13 to inequalities of the form $\mathrm{Pol}_k \mathsf{V} \models u_1 \dots u_n \leq U$, where $U \subseteq A^*$ – see Lemma 4.4). The proof is also similar to the proof of Lemma 3.13.

Lemma 5.3 *Let $k \in \mathbb{N}_0$, $n \in \mathbb{N}$, $n \leq k+1$. Let $(u, U) \in A^* \times 2^{A^*}$ be a k-factorial pair for U. Let $u_1, \dots, u_n \in A^*$ be words such that $u = u_1 \dots u_n$. Then there exist sets of words $U_1, \dots, U_n \in 2^{A^*}$ such that*

- $U \subseteq U_1 \cdots U_n$,
- *for every* $i \in \{1, \ldots, n\}$, *the pair* (u_i, U_i) *is* $\big(k-(n-1)\big)$*-factorial for* U.

Proof We prove the lemma by induction on n. If $n = 1$, the statement is obvious. If $n = 2$, this is by the definition of the k-factoriality. Suppose that $n > 2$. By the assumption, we have $u = u_1 \ldots u_n$. Let $u_1' := u_1 \ldots u_{n-1}$. Then we have $u = u_1' u_n$. By the definition of the k-factoriality, there exist sets $U_1', U_n \in 2^{A^*}$ such that

- $U \subseteq U_1' \cdot U_n$,
- the pairs (u_1', U_1') and (u_n, U_n) are $(k-1)$-factorial for U.

Since we have $u_1' = u_1 \ldots u_{n-1}$, by the induction assumption, there exist sets $U_1, \ldots, U_{n-1} \in 2^{A^*}$ such that

- $U_1' \subseteq U_1 \cdots U_{n-1}$,
- for every $i \in \{1, \ldots, n-1\}$, the pair (u_i, U_i) is $\big(k-1-(n-2)\big)$-factorial for U.

Altogether, we obtain

- $U \subseteq U_1' \cdot U_n \subseteq U_1 \cdots U_{n-1} \cdot U_n$,
- since $k-1-(n-2) = k-(n-1)$, for every $i \in \{1, \ldots, n-1\}$, the pair (u_i, U_i) is $\big(k-(n-1)\big)$-factorial for U,
- the pair (u_n, U_n) is $(k-1)$-factorial for U, hence it is also $\big(k-(n-1)\big)$-factorial for U by Item 3 of Lemma 5.2.

□

5.2 Multiplication on $2^{M \times 2^M}$

Most of the content of this subsection is adopted from the author's PhD thesis [36, 2.3.2]. The connection between the defined multiplication on $2^{M \times 2^M}$ and a lower set monoid was mentioned also in the author's paper [37, Remark 4.8].

Given an arbitrary monoid M, we will use the following special multiplication on the set $2^{M \times 2^M}$:

$$\mathcal{S}_1 \cdot \mathcal{S}_2 = \big\{(s_1 s_2, S) \mid \exists S_1, S_2 \in 2^M : (s_1, S_1) \in \mathcal{S}_1, (s_2, S_2) \in \mathcal{S}_2, S \subseteq S_1 S_2\big\}.$$

When denoting $\mathcal{M} = \big\{\mathcal{S} \cdot \mathcal{T} \mid \mathcal{S}, \mathcal{T} \in 2^{M \times 2^M}\big\}$ and $1_{\mathcal{M}} = \big\{(1, \{1\})\big\} \in 2^{M \times 2^M}$, we obtain a monoid $(\mathcal{M}, \cdot, 1_{\mathcal{M}})$. This is an instance of a *lower set monoid*[8], which was studied, e.g., in [3, 12, 13, 31]. However, despite this interesting connection, we will not need this point of view in this paper.

[8] Specifically, it is the lower set monoid of the ordered monoid $(M \times 2^M, \leq)$, where the corresponding partial order $\leq$ on $M \times 2^M$ is defined in the same way as in (12):

$$(s, S) \leq (t, T) \;\Leftrightarrow\; \big(s = t \wedge S \subseteq T\big).$$

Proposition 5.4 *Let $n \in \mathbb{N}$, $\mathcal{S}_1, \ldots, \mathcal{S}_n \in 2^{M\times 2^M}$. Then*

$$\mathcal{S}_1 \cdots \mathcal{S}_n = \{(s_1 \ldots s_n, S) \mid \exists S_1, \ldots, S_n \in 2^M : S \subseteq S_1 \cdots S_n, \\ (s_i, S_i) \in \mathcal{S}_i, i = 1, \ldots, n\}.$$

Proof Easy. It suffices to use the induction on n. □

Further, note that the whole set $2^{M\times 2^M}$ equipped with this multiplication forms just a semigroup. It is not a monoid for the following reason. Given a set $\mathcal{S} \in 2^{M\times 2^M}$ that contains a pair (s, S) and does not contain a pair (s, S') for a subset $S' \subsetneqq S$, we have $\mathcal{S} \cdot \{(1, \{1\})\} \supsetneqq \mathcal{S}$. This means that the pair $\{(1, \{1\})\}$ is not the identity element. Hence the semigroup $(2^{M\times 2^M}, \cdot)$ does not have the identity element.

The following lemma is analogous to Lemma 4.7. We will need it in Subsection 5.5.

Lemma 5.5 *Let M be a finite monoid, $\mathcal{E}$ be an idempotent element of the semigroup $2^{M\times 2^M}$, and (f, F) be an element of $\mathcal{E}$. Then there exist elements $(f_1, F_1), (f_3, F_3) \in \mathcal{E}$ and $(f_2, F_2) \in \mathcal{E} \cap E(M \times 2^M)$ such that*
$(f, F) \le (f_1, F_1) \cdot (f_2, F_2) \cdot (f_3, F_3)$..

Proof Analogous to the proof of Lemma 4.7 (see [37, Lemma 4.6]). Since we have $(f, F) \in \mathcal{E} = \mathcal{E} \cdot \mathcal{E}$, there exist elements $(g_1, G_1), (g'_1, G'_1) \in \mathcal{E}$ such that $(f, F) \le (g_1, G_1) \cdot (g'_1, G'_1)$. Then, for every $i \in \mathbb{N}$, we successively obtain elements $(g_{i+1}, G_{i+1}), (g'_{i+1}, G'_{i+1}) \in \mathcal{E}$ such that $(g'_i, G'_i) \le (g_{i+1}, G_{i+1}) \cdot (g'_{i+1}, G'_{i+1})$. Since $|\mathcal{E}| \le |M \times 2^M| = |M| \cdot 2^{|M|}$, there exist indices $j < k \le |M| \cdot 2^{|M|} + 1$ such that $(g'_j, G'_j) = (g'_k, G'_k)$. Then we obtain

$$(f, F) \le (g_1, G_1) \cdots (g_j, G_j) \cdot \big((g_{j+1}, G_{j+1}) \cdots (g_k, G_k)\big)^{\omega} \cdot (g'_k, G'_k).$$

It suffices to choose

- $(f_1, F_1) = (g_1, G_1) \cdots (g_j, G_j) \in \mathcal{E}$,
- $(f_2, F_2) = \big((g_{j+1}, G_{j+1}) \cdots (g_k, G_k)\big)^{\omega} \in \mathcal{E} \cap E(M \times 2^M)$,
- $(f_3, F_3) = (g'_k, G'_k) \in \mathcal{E}$.

□

Finally, we show a way how to extend a monoid homomorphism $\alpha : M \to N$ to a semigroup homomorphism $2^{M\times 2^M} \to 2^{N\times 2^N}$. We define

$$\forall \mathcal{S} \in 2^{M\times 2^M} : \quad \alpha(\mathcal{S}) := \big\{\big(\alpha(s), \alpha(S)\big) \mid (s, S) \in \mathcal{S}\big\}.$$

Let's check that the map $\alpha : 2^{M\times 2^M} \to 2^{N\times 2^N}$ defined in this way is indeed a homomorphism. For every pair of elements $\mathcal{S}, \mathcal{T} \in 2^{M\times 2^M}$, we have

$$\alpha(\mathcal{S} \cdot \mathcal{T}) = \big\{\big(\alpha(st), \alpha(R)\big) \mid \exists S, T \in 2^M : (s, S) \in \mathcal{S}, (t, T) \in \mathcal{T},\ R \in 2^M, R \subseteq ST\big\}$$

and

$$\begin{aligned}\alpha(\mathcal{S})\cdot\alpha(\mathcal{T}) &= \big\{\big(\alpha(s)\cdot\alpha(t), P\big) \mid \\ &\qquad \exists S, T \in 2^M : (s,S)\in\mathcal{S}, (t,T)\in\mathcal{T},\ P\in 2^N, P\subseteq \alpha(S)\cdot\alpha(T)\big\} \\ &= \big\{\big(\alpha(st), P\big) \mid \exists S, T\in 2^M : (s,S)\in\mathcal{S}, (t,T)\in\mathcal{T},\ P\in 2^N, P\subseteq\alpha(ST)\big\} \\ &= \big\{\big(\alpha(st), \alpha(R)\big) \mid \exists S, T\in 2^M : (s,S)\in\mathcal{S}, (t,T)\in\mathcal{T},\ R\in 2^M, R\subseteq ST\big\} \\ &= \alpha(\mathcal{S}\cdot\mathcal{T}).\end{aligned}$$

We have shown that $\alpha\colon 2^{M\times 2^M} \to 2^{N\times 2^N}$ is a semigroup homomorphism.

5.3 (k, l)-index of a sequence of words

Now fix again an alphabet A, a finite monoid M, and a surjective homomorphism $\alpha\colon A^* \to M$. Further, fix a locally finite pseudovariety V and a pseudovariety X.

For our proof, we will need the following definition of a (k, l)-*index* of a sequence of words. It is analogous to the definition of a k-index from the previous section (see Subsection 4.1).

Definition 5.6 Let $(u_1, \ldots, u_n) \in (A^*)^n$, where $n \in \mathbb{N}$, be a sequence of words, $k, l \in \mathbb{N}$. For $i = 1, \ldots, n$, let $\mathcal{U}_i$ be the maximal subset of $A^* \times 2^{A^*}$ such that, for every pair $(v_i, V_i) \in \mathcal{U}_i$, the relation $\mathrm{Pol}_k\,\mathsf{V} \models u_i \le v_i$ holds and the pair (v_i, V_i) is l-factorial for X. A (k, l)-*index of the sequence* $(u_1, \ldots, u_n)$ is denoted by $\mathrm{i}_{k,l}(u_1, \ldots, u_n)$ and is defined in the following way:

$$\mathrm{i}_{k,l}(u_1,\ldots,u_n) := \big|\big\{\mathcal{E}\in E(2^{M\times 2^M}) \mid \exists \iota,\kappa,\lambda\in\{1,\ldots,n\}, \iota\le\kappa<\lambda, \\ \kappa-\iota < 2^{|M|\cdot 2^{|M|}} : \alpha(\mathcal{U}_\iota\cdots\mathcal{U}_\kappa)\cdot\mathcal{E} = \alpha(\mathcal{U}_\iota\cdots\mathcal{U}_\kappa), \mathcal{E} = \big(\alpha(\mathcal{U}_{\kappa+1}\cdots\mathcal{U}_\lambda)\big)^\omega\big\}\big|.$$

In other words, the (k, l)-index of a sequence $(u_1, \ldots, u_n)$ counts a number of different idempotents of $2^{M\times 2^M}$ of the form $\big(\alpha(\mathcal{U}_{\kappa+1}\cdots\mathcal{U}_\lambda)\big)^\omega$ that can be inserted into the sequence $\alpha(\mathcal{U}_1), \ldots, \alpha(\mathcal{U}_n)$ at a position κ without changing the product $\alpha(\mathcal{U}_\iota)\cdots\alpha(\mathcal{U}_\kappa)$ for some index $\iota \in \{\kappa - 2^{|M|\cdot 2^{|M|}} + 1, \ldots, \kappa\}$.

Note that $\mathrm{i}_{k,l}(u_1, \ldots, u_n) \in \{0, \ldots, 2^{|M|\cdot 2^{|M|}}\}$.

Corollary 5.7 *Let* $n > 2^{|M|\cdot 2^{|M|}}$ *and* $u_1, \ldots, u_n \in A^*$. *Then* $\mathrm{i}_{k,l}(u_1, \ldots, u_n) \ge 1$.

Proof Analogous to the proof of Corollary 4.3. □

5.4 Factorization of inequalities and l-factorial pairs

Lemmas 5.8 and 5.9 in this subsection are analogous to Lemmas 4.4 and 4.5, respectively, from Subsection 4.2.

Lemma 5.8 *Let* $n, k, l \in \mathbb{N}$, $k, l \ge n-1$, $u_1, \ldots, u_n \in A^*$, $\mathcal{U} \subseteq A^* \times 2^{A^*}$. *Suppose that, for every pair* $(v, V) \in \mathcal{U}$, *we have* $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \ldots u_n \le v$ *and* (v, V) *is* l-*factorial for* X. *For* $i = 1, \ldots, n$, *let* $p_i \in \{n-1, \ldots, k\}$ *and* $q_i \in \{n-1, \ldots, l\}$ *and* *let $\mathcal{U}_i$ be the maximal subset of $A^* \times 2^{A^*}$ such that, for every pair $(v_i, V_i) \in \mathcal{U}_i$, the relation $\mathrm{Pol}_{k-p_i}\, \mathsf{V} \models u_i \leq v_i$ holds and the pair (v_i, V_i) is $(l-q_i)$-factorial for X. Then $\mathcal{U} \subseteq \mathcal{U}_1 \cdots \mathcal{U}_n$.*

Proof The proof is analogous to the proof of Lemma 4.4, although a bit more complicated due to the more complex structure of the given set $\mathcal{U} \subseteq A^* \times 2^{A^*}$ compared to the set $U \subseteq A^*$.

Let $(v, V) \in \mathcal{U}$ be an arbitrary pair. Since $\mathrm{Pol}_k\, \mathsf{V} \models u_1 \dots u_n \leq v$ by the assumption, there exist words $v_1, \dots, v_n \in A^*$ satisfying $v = v_1 \dots v_n$ and $\mathrm{Pol}_{k-(n-1)}\, \mathsf{V} \models u_i \leq v_i$ for $i = 1, \dots, n$ by Lemma 3.13. Since we have $p_i \geq n-1$, i.e., $k-(n-1) \geq k-p_i$, the preceding relation implies that also $\mathrm{Pol}_{k-p_i}\, \mathsf{V} \models u_i \leq v_i$ for $i = 1, \dots, n$.

Further, since the pair $(v_1 \dots v_n, V)$ is l-factorial for X, there exist sets of words $V_1, \dots, V_n \in 2^{A^*}$ such that

i) $V \subseteq V_1 \cdots V_n$,
ii) for every $i \in \{1, \dots, n\}$, the pair (v_i, V_i) is $\big(l-(n-1)\big)$-factorial for X,

by Lemma 5.3. Since we have $q_i \geq n-1$, i.e., $l-(n-1) \geq l-q_i$, Item ii) implies that the pair (v_i, V_i) is also $\big(l-q_i\big)$-factorial for X, for $i = 1, \dots, n$. Then, using Proposition 5.4, we obtain that

$$\begin{aligned}\mathcal{U} &\subseteq \{(v_1 \dots v_n, V) \mid \exists V_1, \dots, V_n \in 2^{A^*} \colon V \subseteq V_1 \cdots V_n,\ (v_i, V_i) \in \mathcal{U}_i, i = 1, \dots, n\} \\ &= \mathcal{U}_1 \cdots \mathcal{U}_n.\end{aligned}$$

□

For the purposes of the following lemma, we denote

$$N := 2^{|M| \cdot 2^{|M|}}.$$

Lemma 5.9 *Let $n \in \mathbb{N}$. Let $u_1, \dots, u_n \in A^*$ be arbitrary words. Let $k, l \in \mathbb{N}$, $k \geq N$, $l \geq N$. Let $\mathcal{U}$ be a subset of $A^* \times 2^{A^*}$ such that, for every pair $(v, V) \in \mathcal{U}$, the relation*

$$\mathrm{Pol}_{k+2\cdot \mathrm{i}_{k-N,l-N}(u_1,\dots,u_n)}\, \mathsf{V} \models u_1 \dots u_n \leq v$$

holds and the pair (v, V) is $\big(l + 2 \cdot \mathrm{i}_{k-N,l-N}(u_1, \dots, u_n)\big)$-factorial for X. For $i = 1, \dots, n$, let $\mathcal{U}_i$ be the maximal subset of $A^* \times 2^{A^*}$ such that, for every pair $(v_i, V_i) \in \mathcal{U}_i$, the relation

$$\mathrm{Pol}_{k-N}\, \mathsf{V} \models u_i \leq v_i$$

holds and the pair (v_i, V_i) is $(l-N)$-factorial for X. Then there exist sets $\mathcal{W}_1, \dots, \mathcal{W}_m \subseteq A^* \times 2^{A^*}$, where $m \in \mathbb{N}$, satisfying the following two conditions:

1. *$\mathcal{U} \subseteq \mathcal{W}_1 \cdots \mathcal{W}_m$,*
2. *for every $i \in \{1, \dots, m\}$, the set $\mathcal{W}_i$ is of the form*
   (i) $\mathcal{W}_i = \mathcal{U}_\iota \cdot \mathcal{U}_{\iota+1} \cdots \mathcal{U}_\kappa$ for some indices $\iota, \kappa \in \{1, \dots, n\}$, $\iota \leq \kappa$, or

(ii) $\mathcal{W}_i = \mathcal{U}_\iota \cdot \mathcal{U}_{\iota+1} \cdots \mathcal{U}_\kappa \cdot \overline{\mathcal{U}} \cdot \mathcal{U}_{\iota'} \cdot \mathcal{U}_{\iota'+1} \cdots \mathcal{U}_{\kappa'}$ *for some set* $\overline{\mathcal{U}} \subseteq A^* \times 2^{A^*}$ *such that, for every pair* $(\overline{v}, \overline{V}) \in \overline{\mathcal{U}}$, *the relation*

$$\mathrm{Pol}_{k-N}\,\mathsf{V} \models u_{\kappa+1} \ldots u_{\iota'-1} \leq \overline{v}$$

holds and the pair $(\overline{v}, \overline{V})$ *is* $(l-N)$*-factorial for* X, *for some indices* $\iota, \kappa, \iota', \kappa' \in \{1, \ldots, n\}$, $\iota \leq \kappa < \iota' \leq \kappa'$ *for which there exists an idempotent* $\mathcal{E} \in E(2^{M \times 2^M})$ *satisfying the relations*

$$\alpha(\mathcal{U}_\iota \cdots \mathcal{U}_\kappa) = \alpha(\mathcal{U}_\iota \cdots \mathcal{U}_\kappa) \cdot \mathcal{E}, \quad \alpha(\mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'}) = \alpha(\mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'}) \cdot \mathcal{E},$$

and

$$\mathcal{E} = \big(\alpha(\mathcal{U}_{\kappa+1} \cdots \mathcal{U}_\lambda)\big)^K$$

for some index $\lambda \in \{\kappa+1, \ldots, n\}$, $\lambda - \kappa \leq N$, *and a number* $K \in \{1, \ldots, N\}$.

Remark 5.10 According to the formulation of Lemma 5.9, it is possible that some of the sets $\mathcal{W}_1, \ldots, \mathcal{W}_m$ are of the form (ii), where $\iota' = \kappa + 1$. In this case, by the notation $u_{\kappa+1} \ldots u_{\iota'-1}$ we mean the empty word ε.

Proof of Lemma 5.9 The proof is analogous to the proof of Lemma 4.5, although a bit more complicated due to the more complex structure of the given set $\mathcal{U} \subseteq A^* \times 2^{A^*}$ compared to the set $U \subseteq A^*$.

The proof goes by induction on the index $\mathrm{i}_{k-N,l-N}(u_1, \ldots, u_n)$. To simplify the notation, we will write $\mathrm{i}(u_1, \ldots, u_n)$ in place of $\mathrm{i}_{k-N,l-N}(u_1, \ldots, u_n)$ throughout this proof.

Recall that we have $N = 2^{|M| \cdot 2^{|M|}}$. If $\mathrm{i}(u_1, \ldots, u_n) = 0$, then $n \leq N$ by Corollary 5.7. By the assumption, for every pair $(v, V) \in \mathcal{U}$, the relation $\mathrm{Pol}_k\,\mathsf{V} \models u_1 \ldots u_n \leq v$ holds and (v, V) is l-factorial for X. Using Lemma 5.8, we obtain that $\mathcal{U} \subseteq \mathcal{U}_1 \cdots \mathcal{U}_n$. It suffices to choose $\mathcal{W}_1 := \mathcal{U}_1 \cdots \mathcal{U}_n$.

Now suppose that $\mathrm{i}(u_1, \ldots, u_n) \geq 1$. If $n \leq N$, we proceed in the same way as in the previous case. Suppose that $n > N$. Then, by Lemma 4.2, there exist indices $1 \leq \kappa < \lambda \leq N+1$ such that $\alpha(\mathcal{U}_1 \cdots \mathcal{U}_\kappa) = \alpha(\mathcal{U}_1 \cdots \mathcal{U}_\kappa) \cdot \big(\alpha(\mathcal{U}_{\kappa+1} \cdots \mathcal{U}_\lambda)\big)^\omega$. Let

$$\mathcal{E} := \big(\alpha(\mathcal{U}_{\kappa+1} \cdots \mathcal{U}_\lambda)\big)^\omega.$$

Since the monoid $2^{M \times 2^M}$ has size N, there exists an integer $K \in \{1, \ldots, N\}$ such that

$$\mathcal{E} = \big(\alpha(\mathcal{U}_{\kappa+1} \cdots \mathcal{U}_\lambda)\big)^\omega = \big(\alpha(\mathcal{U}_{\kappa+1} \cdots \mathcal{U}_\lambda)\big)^K.$$

Recall that, by the assumptions, for every pair $(v, V) \in \mathcal{U}$, the relation

$$\mathrm{Pol}_{k+2\cdot\mathrm{i}(u_1,\ldots,u_n)}\,\mathsf{V} \models u \leq v$$

holds and the pair (v, V) is $\big(l + 2 \cdot \mathrm{i}(u_1, \ldots, u_n)\big)$-factorial for X and, for $i = 1, \ldots, n$, the set $\mathcal{U}_i$ is defined as

$$\mathcal{U}_i = \{(v_i, V_i) \in A^* \times 2^{A^*} \mid \mathrm{Pol}_{k-N}\,\mathsf{V} \models u_i \le v_i,\ (v_i, V_i) \text{ is } (l-N)\text{-factorial for } \mathsf{X}\}.$$

Let

$$\mathcal{W}_0 := \{(w_0, W_0) \in A^* \times 2^{A^*} \mid \mathrm{Pol}_{k-N+\kappa-1}\,\mathsf{V} \models u_1 \dots u_\kappa \le w_0,$$
$$(w_0, W_0) \text{ is } (l-N+\kappa-1)\text{-factorial for } \mathsf{X}\}.$$

By Lemma 5.8, we obtain that $\mathcal{W}_0 \subseteq \mathcal{U}_1 \dots \mathcal{U}_\kappa$. Further, we have

$$k - N + \kappa - 1 \le k - N + N - 1 = k - 1 < k + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 1$$

and similarly

$$l - N + \kappa - 1 < l + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 1.$$

Let

$$\mathcal{W}' := \{(w', W') \in A^* \times 2^{A^*} \mid \mathrm{Pol}_{k+2\cdot \mathrm{i}(u_1,\dots,u_n)-1}\,\mathsf{V} \models u_{\kappa+1} \dots u_n \le w',$$
$$(w', W') \text{ is } (l + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 1)\text{-factorial for } \mathsf{X}\}.$$

Then, by Lemma 5.8, we obtain $\mathcal{U} \subseteq \mathcal{W}_0\mathcal{W}' \subseteq \mathcal{U}_1 \cdots \mathcal{U}_\kappa \cdot \mathcal{W}'$.

If $\mathrm{i}(u_{\kappa+1}, \dots, u_n) < \mathrm{i}(u_1, \dots, u_n)$, since

$$k + 2 \cdot \mathrm{i}(u_{\kappa+1}, \dots, u_n) \le k + 2 \cdot \big(\mathrm{i}(u_1, \dots, u_n) - 1\big) = k + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 2 <$$
$$< k + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 1,$$

and similarly

$$l + 2 \cdot \mathrm{i}(u_{\kappa+1}, \dots, u_n) < l + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 1,$$

we obtain that, for every pair $(w', W') \in \mathcal{W}'$, the relation

$$\mathrm{Pol}_{k+2\cdot \mathrm{i}(u_{\kappa+1},\dots,u_n)}\,\mathsf{V} \models u_{\kappa+1} \dots u_n \le w'$$

holds and the pair (w', W') is $(l + 2 \cdot \mathrm{i}(u_{\kappa+1}, \dots, u_n))$-factorial for X. Then, by the induction assumption, there exist sets $\mathcal{W}'_1, \dots, \mathcal{W}'_{m'} \subseteq A^* \times 2^{A^*}$, where $m' \in \mathbb{N}$, satisfying the following two conditions:

1. $\mathcal{W}' \subseteq \mathcal{W}'_1 \cdots \mathcal{W}'_{m'}$,
2. for every $i \in \{1, \dots, m'\}$, the set $\mathcal{W}'_i$ is of the form (i) or (ii).

If we choose $\mathcal{W}_1 = \mathcal{U}_1 \cdots \mathcal{U}_\kappa$, we obtain $\mathcal{U} \subseteq \mathcal{W}_1 \cdot \mathcal{W}'_1 \cdots \mathcal{W}'_{m'}$.

Now suppose that $\mathrm{i}(u_{\kappa+1}, \dots, u_n) = \mathrm{i}(u_1, \dots, u_n)$. Then, by the definition of $\mathrm{i}(u_{\kappa+1}, \dots, u_n)$, there exist indices ι', κ', $\kappa + 1 \le \iota' \le \kappa' < n$, $\kappa' - \iota' < N$ such that

$$\alpha(\mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'}) = \alpha(\mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'}) \cdot \mathcal{E}.$$

Choose the biggest possible such ι' and some κ' for this ι'. Then we have

$$\mathrm{i}(u_{\kappa'+1}, \dots, u_n) < \mathrm{i}(u_1, \dots, u_n).$$

Hence

$$k + 2 \cdot \mathrm{i}(u_{\kappa'+1}, \dots, u_n) \leq k + 2 \cdot \big(\mathrm{i}(u_1, \dots, u_n) - 1\big) = k + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 2$$

and similarly

$$l + 2 \cdot \mathrm{i}(u_{\kappa'+1}, \dots, u_n) \leq l + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 2.$$

Recall that, for every pair $(w', W') \in \mathcal{W}'$, the relation

$$\mathrm{Pol}_{k+2\cdot\mathrm{i}(u_1,\dots,u_n)-1} \mathsf{V} \models u_{\kappa+1} \dots u_n \leq w'$$

holds and the pair (w', W') is $(l + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 1)$-factorial for X. Let

$$\overline{\mathcal{W}} := \{(\overline{w}, \overline{W}) \in A^* \times 2^{A^*} \mid \mathrm{Pol}_{k+2\cdot\mathrm{i}(u_1,\dots,u_n)-2} \mathsf{V} \models u_{\kappa+1} \dots u_{\kappa'} \leq \overline{w},$$
$$(\overline{w}, \overline{W}) \text{ is } (l + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 2)\text{-factorial for } \mathsf{X}\},$$

$$\mathcal{W}'' := \{(w'', W'') \in A^* \times 2^{A^*} \mid \mathrm{Pol}_{k+2\cdot\mathrm{i}(u_{\kappa'+1},\dots,u_n)} \mathsf{V} \models u_{\kappa'+1} \dots u_n \leq w'',$$
$$(w'', W'') \text{ is } (l + 2 \cdot \mathrm{i}(u_{\kappa'+1}, \dots, u_n))\text{-factorial for } \mathsf{X}\}.$$

By Lemma 5.8, we obtain that $\mathcal{W}' \subseteq \overline{\mathcal{W}}\mathcal{W}''$. This implies that

$$\mathcal{U} \subseteq \mathcal{U}_1 \cdots \mathcal{U}_\kappa \cdot \mathcal{W}' \subseteq \mathcal{U}_1 \cdots \mathcal{U}_\kappa \cdot \overline{\mathcal{W}} \cdot \mathcal{W}''.$$

Further, by the induction assumption, there exist sets $\mathcal{W}''_1, \dots, \mathcal{W}''_{m''} \subseteq A^* \times 2^{A^*}$, where $m'' \in \mathbb{N}$, satisfying the following two conditions:

1. $\mathcal{W}'' \subseteq \mathcal{W}''_1 \cdots \mathcal{W}''_{m''}$,
2. for every $i \in \{1, \dots, m''\}$, the set $\mathcal{W}''_i$ is of the form (i) or (ii).

Then we obtain

$$\mathcal{U} \subseteq \mathcal{U}_1 \cdots \mathcal{U}_\kappa \cdot \overline{\mathcal{W}} \cdot \mathcal{W}'' \subseteq \mathcal{U}_1 \cdots \mathcal{U}_\kappa \cdot \overline{\mathcal{W}} \cdot \mathcal{W}''_1 \cdots \mathcal{W}''_{m''}.$$

By the definition of $\overline{\mathcal{W}}$, for every pair $(\overline{w}, \overline{W}) \in \overline{\mathcal{W}}$, the relation

$$\mathrm{Pol}_{k+2\cdot\mathrm{i}(u_1,\dots,u_n)-2} \mathsf{V} \models u_{\kappa+1} \dots u_{\kappa'} \leq \overline{w}$$

holds and the pair $(\overline{w}, \overline{W})$ is $(l + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 2)$-factorial for X. Further, recall that, for every $i \in \{\kappa + 1, \dots, \kappa'\}$, we have

$$\mathcal{U}_i = \{(v_i, V_i) \in A^* \times 2^{A^*} \mid \mathrm{Pol}_{k-N} \mathsf{V} \models u_i \leq v_i, (v_i, V_i) \text{ is } (l-N)\text{-factorial for } \mathsf{X}\}.$$

Let

$$\overline{\mathcal{U}} := \{(\overline{v}, \overline{V}) \in A^* \times 2^{A^*} \mid \mathrm{Pol}_{k-N} \mathsf{V} \models u_{\kappa+1} \dots u_{\iota'-1} \leq \overline{v}, \\ (\overline{v}, \overline{V}) \text{ is } (l-N)\text{-factorial for } \mathsf{X}\},$$

where we put $u_{\kappa+1} \dots u_{\iota'-1} := \varepsilon$ in the case when $\iota' = \kappa + 1$ (as in Remark 5.10). Since $\kappa' - \iota' < N$ and

$$k + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 2 - N \geq k + 2 - 2 - N = k - N$$

and similarly

$$l + 2 \cdot \mathrm{i}(u_1, \dots, u_n) - 2 - N \geq l - N,$$

we obtain that $\overline{\mathcal{W}} \subseteq \overline{\mathcal{U}} \cdot \mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'}$ by Lemma 5.8. Finally, if we choose

$$\mathcal{W}_1 := \mathcal{U}_1 \cdots \mathcal{U}_\kappa \cdot \overline{\mathcal{U}} \cdot \mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'},$$

we obtain

$$\mathcal{U} \subseteq \mathcal{U}_1 \cdots \mathcal{U}_\kappa \cdot \overline{\mathcal{W}} \cdot \mathcal{W}''_1 \cdots \mathcal{W}''_{m''} \subseteq \mathcal{U}_1 \cdots \mathcal{U}_\kappa \cdot \overline{\mathcal{U}} \cdot \mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'} \cdot \mathcal{W}''_1 \cdots \mathcal{W}''_{m''} = \\ = \mathcal{W}_1 \cdot \mathcal{W}''_1 \cdots \mathcal{W}''_{m''},$$

where $\mathcal{W}_1$ and all the $\mathcal{W}''_i$ satisfy Condition 2 of Lemma 5.9. The proof of Lemma 5.9 has been finished. □

5.5 Properties of ω-reducible pairs

Let U be an arbitrary pseudovariety of ordered monoids. Recall that we have fixed a pseudovariety X, an alphabet A, a finite monoid M, and a surjective homomorphism $\alpha \colon A^* \to M$.

A pair $(s, \mathcal{S}) \in M \times 2^{M \times 2^M}$ is called *ω-reducible for* (U, X) if there exists an ω-word $u \in \widehat{\alpha}^{-1}(s)$ such that, for every $(t, T) \in \mathcal{S} \subseteq M \times 2^M$, there exists an ω-word $v \in \widehat{\alpha}^{-1}(t)$ having the property that $\mathsf{U} \models u \leq v$ and such that, for every $q \in T$, there exists an ω-word $w \in \widehat{\alpha}^{-1}(q)$ satisfying $\mathsf{X} \models v \leq w$.

Recall that we have fixed also a locally finite pseudovariety V. In this subsection, we employ Lemma 5.9 to prove the ω-reducibility for $(\mathrm{Pol}\, \mathsf{V}, \mathsf{X})$ of pairs $(s, \mathcal{S}) \in M \times 2^{M \times 2^M}$ of the form $(s, \mathcal{S}) = \big(e, \alpha(\mathcal{U})\big)$ assuming that the pairs $\big(e, \alpha(\mathcal{U}_1)\big), \dots, \big(e, \alpha(\mathcal{U}_n)\big)$ are ω-reducible for $(\mathrm{Pol}\, \mathsf{V}, \mathsf{X})$, where $e \in E(M)$, $\alpha(u_1) = \alpha(u_2) = \cdots = \alpha(u_n) = e$, and $u_1, u_2, \dots, u_n \in A^*$ and $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n \subseteq A^* \times 2^{A^*}$ are from Lemma 5.9. Moreover, adding an extra condition, we are able to prove the ω-reducibility of these pairs also for $(\mathrm{Pol}\, \mathsf{U}, \mathsf{X})$, where U is an *arbitrary* pseudovariety.

5.5.1 Basic properties

We begin with simpler properties of ω-reducible pairs $(s, \mathcal{S}) \in M \times 2^{M \times 2^M}$, which we use in the proof of our target claim.

The following lemmas were proven in [37, Lemma 4.10] for the case $\mathsf{V} = \mathrm{Pol}\,\mathsf{W}$, $\mathsf{X} = \mathrm{Pol}\,\mathsf{V}^{\mathrm{d}}$, where W is a locally finite selfdual pseudovariety. The proofs for the general case are analogous. We present general proofs here except for the first two (Lemmas 5.11 and 5.12), which are obvious.

Lemma 5.11 *Let* U *be an arbitrary pseudovariety. Let* $s \in M$ *be an arbitrary element. The pair* $\big(s, \{(s, \{s\})\}\big)$ *is* ω*-reducible for* (U, X).

Lemma 5.12 *Let* U *be an arbitrary pseudovariety. Let* $(s, \mathcal{S})$ *be an* ω*-reducible pair for* (U, X) *and let* $\mathcal{S}' \subseteq \mathcal{S}$. *Then the pair* $(s, \mathcal{S}')$ *is also* ω*-reducible for* (U, X).

Lemma 5.13 *Let* U *be an arbitrary pseudovariety. Let* $(s_1, \mathcal{S}_1)$, $(s_2, \mathcal{S}_2)$ *be* ω*-reducible pairs for* (U, X). *Then the pair* $(s_1 s_2, \mathcal{S}_1\mathcal{S}_2)$ *is also* ω*-reducible for* (U, X).

Proof Analogous to the proof of Lemma 4.10. Suppose that the pairs $(s_1, \mathcal{S}_1)$, $(s_2, \mathcal{S}_2)$ are ω-reducible for (U, X). This means that there exist ω-words $x_1, x_2 \in \Omega_A^{\omega}\mathsf{M}$ and sets $\mathcal{X}_1, \mathcal{X}_2 \subseteq \Omega_A^{\omega}\mathsf{M} \times 2^{\Omega_A^{\omega}\mathsf{M}}$ such that the following conditions are satisfied for $i \in \{1, 2\}$:

i) for every pair $(y_i, Y_i) \in \mathcal{X}_i$, the relations $\mathsf{U} \models x_i \leq y_i$, $\mathsf{X} \models y_i \leq Y_i$ hold,
ii) $\widehat{\alpha}(x_i) = s_i$, $\widehat{\alpha}(\mathcal{X}_i) = \mathcal{S}_i$.

We need to show that the pair $(s_1 s_2, \mathcal{S}_1\mathcal{S}_2)$ is also ω-reducible for (U, X), i.e., we need to find an ω-word $x \in \Omega_A^{\omega}\mathsf{M}$ and a set $\mathcal{X} \subseteq \Omega_A^{\omega}\mathsf{M} \times 2^{\Omega_A^{\omega}\mathsf{M}}$ such that the following conditions are satisfied:

i) for every pair $(y, Y) \in \mathcal{X}$, the relations $\mathsf{U} \models x \leq y$, $\mathsf{X} \models y \leq Y$ hold,
ii) $\widehat{\alpha}(x) = s_1 s_2$, $\widehat{\alpha}(\mathcal{X}) = \mathcal{S}_1\mathcal{S}_2$.

It suffices to choose $x := x_1 x_2$, $\mathcal{X} := \mathcal{X}_1\mathcal{X}_2$. Indeed, we have:

i) for every pair $(y, Y) \in \mathcal{X} = \mathcal{X}_1\mathcal{X}_2$, there exist pairs $(y_1, Y_1) \in \mathcal{X}_1$, $(y_2, Y_2) \in \mathcal{X}_2$ such that $y = y_1 y_2$ and $Y \subseteq Y_1 Y_2$; then the relations $\mathsf{U} \models x = x_1 x_2 \leq y_1 y_2 = y$, $\mathsf{X} \models y = y_1 y_2 \leq Y_1 Y_2 \supseteq Y$ hold,
ii) $\widehat{\alpha}(x) = \widehat{\alpha}(x_1)\widehat{\alpha}(x_2) = s_1 s_2$, $\widehat{\alpha}(\mathcal{X}) = \widehat{\alpha}(\mathcal{X}_1)\widehat{\alpha}(\mathcal{X}_2) = \mathcal{S}_1\mathcal{S}_2$.

□

To get a simple notation, we use quadruples $(s, S, t, \mathcal{T})$ and $(\mathsf{U}_0, \mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3)$ in the following definition. Another possibility would be to use more complex ordered sets, describing relations between individual elements, instead. A visualization of these relations is depicted in Figure 1.

Definition 5.14 Let $\mathsf{U}_0, \mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3$ be arbitrary pseudovarieties. We say that a quadruple $(s, S, t, \mathcal{T}) \in M \times 2^M \times M \times 2^{M \times 2^M}$ is *ω-reducible for* $(\mathsf{U}_0, \mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3)$ if there exist ω-words $u, v \in \Omega_A^{\omega}\mathsf{M}$, a set of ω-words $U \subseteq \Omega_A^{\omega}\mathsf{M}$, and a set $\mathcal{V} \subseteq \Omega_A^{\omega}\mathsf{M} \times 2^{\Omega_A^{\omega}\mathsf{M}}$ satisfying the following conditions:

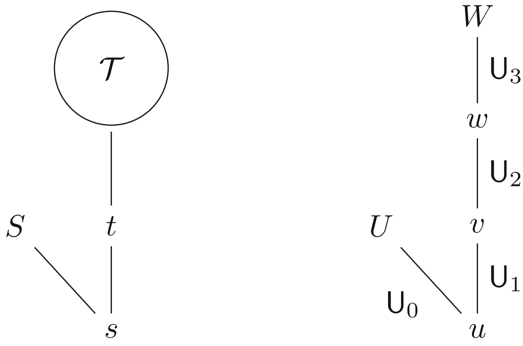

Fig. 1 Scheme of the ω-reducibility of a quadruple $(s, S, t, \mathcal{T})$ for $(\mathsf{U}_0, \mathsf{U}_1, \mathsf{U}_2, \mathsf{U}_3)$ from Definition 5.14

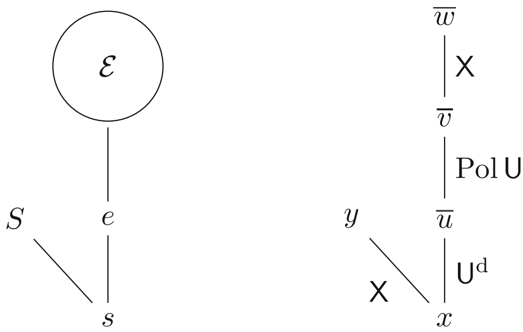

Fig. 2 Scheme of the ω-reducibility of the quadruple $(s, S, e, \mathcal{E})$ for $(\mathsf{X}, \mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U}, \mathsf{X})$ for the proof of Lemma 5.15

i) $\mathsf{U}_0 \models u \leq U$, $\mathsf{U}_1 \models u \leq v$ and, for every pair $(w, W) \in \mathcal{V}$, $\mathsf{U}_2 \models v \leq w$, $\mathsf{U}_3 \models w \leq W$,
ii) $\widehat{\alpha}(u) = s, \widehat{\alpha}(U) = S, \widehat{\alpha}(v) = t, \widehat{\alpha}(\mathcal{V}) = \mathcal{T}$.

Lemma 5.15 *Let* U *be an arbitrary pseudovariety. Let* $(s, S, e, \mathcal{E}) \in M \times 2^M \times E(M) \times E\big(2^{M \times 2^M}\big)$ *be an ω-reducible quadruple for* $(\mathsf{X}, \mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U}, \mathsf{X})$. *Then the pair* $\Big(e, \mathcal{E} \cdot \big\{(s, S), (1, \{1\})\big\} \cdot \mathcal{E}\Big)$ *is ω-reducible for* $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$.

Proof We will proceed analogously to the proof of Lemma 4.12. Let $(s, S, e, \mathcal{E}) \in M \times 2^M \times E(M) \times E\big(2^{M \times 2^M}\big)$ be an ω-reducible quadruple for $(\mathsf{X}, \mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U}, \mathsf{X})$. For a visualization of relations between elements $s, S, e, \mathcal{E}$, using the notation in accordance with the following paragraph, see Figure 2.

Then there exist ω-words $x \in \widehat{\alpha}^{-1}(s)$, $\overline{u} \in \widehat{\alpha}^{-1}(e)$ satisfying $\mathsf{U}^{\mathrm{d}} \models x \leq \overline{u}$ and such that, for every $t \in S$, there exists an ω-word $y \in \widehat{\alpha}^{-1}(t)$ satisfying $\mathsf{X} \models x \leq y$, for every pair $(f, F) \in \mathcal{E} \subseteq M \times 2^M$, there exists an ω-word $\overline{v} \in \widehat{\alpha}^{-1}(f)$ having the property $\mathrm{Pol}\,\mathsf{U} \models \overline{u} \leq \overline{v}$ and such that, for every $h \in F$, there exists an ω-word $\overline{w} \in \widehat{\alpha}^{-1}(h)$ satisfying $\mathsf{X} \models \overline{v} \leq \overline{w}$.

To prove the ω-reducibility of the pair $\Big(e, \mathcal{E} \cdot \big\{(s, S), (1, \{1\})\big\} \cdot \mathcal{E}\Big)$ for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$, we use the description of the pseudovariety $\mathrm{Pol}\,\mathsf{U}$ by pseudoinequalities from Proposition 2.2:

$$\mathrm{Pol}\,\mathsf{U} = [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid u, v \in \overline{\Omega}_A \mathsf{M} \text{ for some } A,\ \mathsf{U} \models u \leq v]\!]. \tag{13}$$

Let $(p, P) \in \mathcal{E} \cdot \{(s, S), (1, \{1\})\} \cdot \mathcal{E} \subseteq M \times 2^M$ be an arbitrary element. Then, by Proposition 5.4, there exist pairs $(f, F), (g, G) \in \mathcal{E}$ and a pair $(s^\star, S^\star) \in \{(s, S), (1, \{1\})\}$ such that $p = fs^\star g$, $P \subseteq FS^\star G$. The set $\mathcal{E}$ is an idempotent of $2^{M\times 2^M}$, hence, by Lemma 5.5, there exist elements

$$(f_1, F_1), (f_3, F_3), (g_1, G_1), (g_3, G_3) \in \mathcal{E}, \quad (f_2, F_2), (g_2, G_2) \in \mathcal{E} \cap E(M \times 2^M)$$

such that the inequalities

$$(f, F) \leq (f_1, F_1) \cdot (f_2, F_2) \cdot (f_3, F_3), \quad (g, G) \leq (g_1, G_1) \cdot (g_2, G_2) \cdot (g_3, G_3)$$

hold. By the ω-reducibility of $(s, S, e, \mathcal{E})$ for $(\mathsf{X}, \mathsf{U}^{\mathrm{d}}, \operatorname{Pol}\mathsf{U}, \mathsf{X})$, for every $t \in S$ there exists an ω-word $y \in \widehat{\alpha}^{-1}(t)$ satisfying $\mathsf{X} \models x \leq y$ and there exist ω-words $v_i \in \widehat{\alpha}^{-1}(f_i)$, $\overline{v_i} \in \widehat{\alpha}^{-1}(g_i)$ satisfying $\operatorname{Pol}\mathsf{U} \models \overline{u} \leq v_i$, $\operatorname{Pol}\mathsf{U} \models \overline{u} \leq \overline{v_i}$ and such that, for all $h_i \in F_i, \overline{h_i} \in G_i$, there exist ω-words $w_i \in \widehat{\alpha}^{-1}(h_i)$, $\overline{w_i} \in \widehat{\alpha}^{-1}(\overline{h_i})$ satisfying $\mathsf{X} \models v_i \leq w_i$, $\mathsf{X} \models \overline{v}_i \leq \overline{w}_i$ for $i = 1, 2, 3$.

Since we have $\mathsf{U} \subseteq \operatorname{Pol}\mathsf{U}$, the preceding properties imply that $\mathsf{U} \models \overline{u} \leq v_i$, $\mathsf{U} \models \overline{u} \leq \overline{v}_i$ for $i = 1, 2, 3$. Further, recall that we have $\mathsf{U}^{\mathrm{d}} \models x \leq \overline{u}$, i.e., $\mathsf{U} \models \overline{u} \leq x$. Then we obtain relations $\mathsf{U} \models \overline{u}\,\overline{u}\,\overline{u} \leq v_3 x \overline{v}_1$ and $\mathsf{U} \models \overline{u}\,\overline{u}\,\overline{u} \leq v_2 v_3 \overline{v}_1$, which imply that

$$\operatorname{Pol}\mathsf{U} \models (\overline{u}\,\overline{u}\,\overline{u})^{\omega+1} \leq (\overline{u}\,\overline{u}\,\overline{u})^{\omega}(v_3 x \overline{v}_1)(\overline{u}\,\overline{u}\,\overline{u})^{\omega}$$

and

$$\operatorname{Pol}\mathsf{U} \models (\overline{u}\,\overline{u}\,\overline{u})^{\omega+1} \leq (\overline{u}\,\overline{u}\,\overline{u})^{\omega}(v_2 v_3 \overline{v}_1)(\overline{u}\,\overline{u}\,\overline{u})^{\omega},$$

respectively, by Relation (13). From these relations, we obtain

$$\operatorname{Pol}\mathsf{U} \models (\overline{u})^{\omega+3} \leq (\overline{u})^{\omega}(v_3 x \overline{v}_1)(\overline{u})^{\omega} \leq (v_2)^{\omega} v_3 x \overline{v}_1 (\overline{v_2})^{\omega} \tag{14}$$

and

$$\operatorname{Pol}\mathsf{U} \models (\overline{u})^{\omega+3} \leq (\overline{u})^{\omega}(v_2 v_3 \overline{v}_1)(\overline{u})^{\omega} \leq (v_2)^{\omega+1} v_3 \overline{v}_1 (\overline{v_2})^{\omega}, \tag{15}$$

respectively.

Further, fix an element

$$q = h_1 h_2 h_3 t^\star \overline{h_1}\,\overline{h_2}\,\overline{h_3} \in P \subseteq FS^\star G \subseteq F_1 F_2 F_3 S^\star G_1 G_2 G_3 \subseteq M,$$

where $h_1 \in F_1, h_2 \in F_2, h_3 \in F_3, t^\star \in S^\star, \overline{h_1} \in G_1, \overline{h_2} \in G_2, \overline{h_3} \in G_3$. The sets F_2 and G_2 are idempotents of 2^M, hence, by Lemma 4.7, there exist elements $h_4, h_6 \in F_2$, $h_5 \in F_2 \cap E(M)$ and $\overline{h_4}, \overline{h_6} \in G_2$, $\overline{h_5} \in G_2 \cap E(M)$ such that $h_2 = h_4 h_5 h_6$ and $\overline{h_2} = \overline{h_4}\,\overline{h_5}\,\overline{h_6}$.

Now choose the relevant ω-words $w_i \in \widehat{\alpha}^{-1}(h_i), \overline{w_i} \in \widehat{\alpha}^{-1}(\overline{h_i})$ for $i = 1, 3, 4, 5, 6$, having the properties $\mathsf{X} \models v_i \leq w_i$, $\mathsf{X} \models \overline{v}_i \leq \overline{w}_i$ for $i = 1, 3$, $\mathsf{X} \models v_2 \leq w_i$, $\mathsf{X} \models \overline{v}_2 \leq \overline{w}_i$ for $i = 4, 5, 6$. Furthermore, if $(s^\star, S^\star) = (s, S)$, choose the relevant

ω-word $y \in \widehat{\alpha}^{-1}(t^\star)$ satisfying $\mathsf{X} \models x \le y$. Then we obtain

$$\mathsf{X} \models v_1(v_2)^{\omega+2}(v_3 x \overline{v_1})(\overline{v_2})^{\omega+2}\overline{v_3} = v_1\, v_2\, (v_2)^\omega\, v_2\, (v_3\, x\, \overline{v_1})\, \overline{v_2}\, (\overline{v_2})^\omega\, \overline{v_2}\, \overline{v_3} \le \\ \le w_1 w_4 (w_5)^\omega w_6 (w_3 y \overline{w_1}) \overline{w_4} (\overline{w_5})^\omega \overline{w_6}\, \overline{w_3}. \quad (16)$$

and

$$\mathsf{X} \models v_1(v_2)^{\omega+3}(v_3 \overline{v}_1)(\overline{v_2})^{\omega+2}\overline{v_3} = v_1\, v_2\, (v_2)^{\omega+1}\, (v_2 v_3 \overline{v}_1)\, \overline{v_2}\, (\overline{v_2})^\omega\, \overline{v_2}\, \overline{v_3} \le \\ \le w_1 w_4 (w_5)^{\omega+1} (w_6 w_3 \overline{w}_1) \overline{w_4} (\overline{w_5})^\omega \overline{w_6}\, \overline{w_3}. \quad (17)$$

We put

$$u = (\overline{u})^{\omega+9},$$
$$v = \begin{cases} v_1(v_2)^{\omega+2} v_3\, x\, \overline{v_1}(\overline{v_2})^{\omega+2}\overline{v_3} & \text{if } (s^\star, S^\star) = (s, S), \\ v_1(v_2)^{\omega+3}\, v_3\;\; \overline{v}_1\, (\overline{v_2})^{\omega+2}\overline{v_3} & \text{if } (s^\star, S^\star) = \big(1, \{1\}\big), \end{cases}$$
$$w = \begin{cases} w_1 w_4 (w_5)^\omega w_6 w_3\; y\; \overline{w_1}\, \overline{w_4}(\overline{w_5})^\omega \overline{w_6}\, \overline{w_3} & \text{if } (s^\star, S^\star) = (s, S), \\ w_1 w_4 (w_5)^{\omega+1} w_6 w_3\, \overline{w}_1\, \overline{w_4}(\overline{w_5})^\omega \overline{w_6}\, \overline{w_3} & \text{if } (s^\star, S^\star) = \big(1, \{1\}\big). \end{cases}$$

Then u, v, w are ω-words satisfying

- $\widehat{\alpha}(u) = e^{\omega+9} = e$,
- $\widehat{\alpha}(v) = \begin{cases} f_1(f_2)^{\omega+2} f_3 s g_1 (g_2)^{\omega+2} g_3 = f_1 f_2 f_3 s g_1 g_2 g_3 = f s g = p & \text{if } (s^\star, S^\star) = (s, S), \\ f_1(f_2)^{\omega+3} f_3\; g_1 (g_2)^{\omega+2} g_3 = f_1 f_2 f_3\; g_1 g_2 g_3 = f g = p & \text{if } (s^\star, S^\star) = \big(1, \{1\}\big). \end{cases}$
-

$$\widehat{\alpha}(w) = h_1 h_4 (h_5)^\omega h_6\, h_3\, t^\star\, \overline{h_1}\;\, \overline{h_4}(\overline{h_5})^\omega \overline{h_6}\, \overline{h_3} = \\ = h_1 (h_4 h_5 h_6)\;\; h_3\, t^\star\, \overline{h_1}\, (\overline{h_4}\, \overline{h_5}\, \overline{h_6})\;\; \overline{h_3} = \\ = h_1 \quad h_2 \quad h_3\, t^\star\, \overline{h_1} \quad \overline{h_2} \quad \overline{h_3} = q,$$

- using Properties (14) and (15):

$$\mathrm{Pol}\,\mathsf{U} \models u = (\overline{u})^{\omega+9} = \overline{u}\;\overline{u}\;\overline{u} \quad (\overline{u})^{\omega+3} \quad \overline{u}\;\overline{u}\;\overline{u} \le \\ \le v_1 v_2 v_2 \quad (v_2)^\omega v_3\, x\, \overline{v_1}(\overline{v_2})^\omega \quad \overline{v_2}\,\overline{v_2}\,\overline{v_3} = \\ = v_1 \quad (v_2)^{\omega+2} v_3\, x\, \overline{v_1}(\overline{v_2})^{\omega+2} \quad \overline{v_3} = v,$$

if $(s^\star, S^\star) = (s, S)$, and

$$\mathrm{Pol}\,\mathsf{U} \models u = (\overline{u})^{\omega+9} = \overline{u}\;\overline{u}\;\overline{u} \quad (\overline{u})^{\omega+3} \quad \overline{u}\;\overline{u}\;\overline{u} \le \\ \le v_1 v_2 v_2 (v_2)^{\omega+1}\; v_3 \overline{v_1}\; (\overline{v_2})^\omega \quad \overline{v_2}\,\overline{v_2}\,\overline{v_3} = \\ = v_1 \quad (v_2)^{\omega+3}\; v_3 \overline{v_1}\; (\overline{v_2})^{\omega+2} \quad \overline{v_3} = v,$$

if $(s^\star, S^\star) = \big(1, \{1\}\big)$,

- $\mathsf{X} \models v \leq w$ by Properties (16) and (17),

with u independent of the choice of an element $(p, P) \in \left(e, \mathcal{E} \cdot \{(s, S), (1, \{1\})\} \cdot \mathcal{E}\right)$ and v independent of the choice of an element $q \in P$.

Altogether, we have proven that the pair $\left(e, \mathcal{E} \cdot \{(s, S), (1, \{1\})\} \cdot \mathcal{E}\right)$ is ω-reducible for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$, as was required. □

To simplify the notation in proofs, we define a binary relation $\leq$ on the set $M \times 2^{M \times 2^M}$ (analogous to the relation defined in (7)):

$$(s, \mathcal{S}) \leq (t, \mathcal{T}) \quad \Leftrightarrow \quad s = t \text{ and } \mathcal{S} \subseteq \mathcal{T}.$$

It is obviously a partial order on $M \times 2^{M \times 2^M}$.

The following corollary is analogous to Corollary 4.13.

Corollary 5.16 *Let* U *be an arbitrary pseudovariety. Let* $\mathcal{S} \in 2^{M \times 2^M}$, $e \in E(M)$, $\mathcal{E} \in E\left(2^{M \times 2^M}\right)$. *If, for every pair* $(s, S) \in \mathcal{S}$, *the quadruple* $(s, S, e, \mathcal{E})$ *is* ω*-reducible for* $(\mathsf{X}, \mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U}, \mathsf{X})$*, then the pair* $(e, \mathcal{E}\mathcal{S}\mathcal{E})$ *is* ω*-reducible for* $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$.

Proof Analogous to the proof of Corollary 4.13. By Lemma 5.15, for every pair $(s, S) \in \mathcal{S}$, the pair $\left(e, \mathcal{E} \cdot \{(s, S), (1, \{1\})\} \cdot \mathcal{E}\right)$ is ω-reducible for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$. Recall that $\{(1, \{1\})\}$ is the neutral element of the monoid $\mathcal{M} = \{\mathcal{S} \cdot \mathcal{T} \mid \mathcal{S}, \mathcal{T} \in 2^{M \times 2^M}\}$ introduced in Section 5.2. Since $\mathcal{E} = \mathcal{E} \cdot \mathcal{E}$, we have $\mathcal{E} \in \mathcal{M}$. Hence $\mathcal{E} \subseteq \mathcal{E} \cdot \{(s, S), (1, \{1\})\} \cdot \mathcal{E}$ for arbitrary $(s, S) \in \mathcal{S}$. Then we have

$$(e, \mathcal{E}\mathcal{S}\mathcal{E}) \leq \left(e, \prod_{(s,S)\in\mathcal{S}} \left(\mathcal{E}\cdot\{(s, S), (1, \{1\})\}\cdot\mathcal{E}\right)\right) = \prod_{(s,S)\in\mathcal{S}} \left(e, \mathcal{E}\cdot\{(s, S), (1, \{1\})\}\cdot\mathcal{E}\right).$$

Hence, the pair $(e, \mathcal{E}\mathcal{S}\mathcal{E})$ is ω-reducible for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$ by Lemmas 5.12 and 5.13. □

Now we get back to our fixed *locally finite* pseudovariety V. Recall that we have fixed also a surjective homomorphism $\alpha\colon A^* \to M$ into a finite monoid M. For the following corollary, we need to assume additionally that the homomorphism α is V-*compatible*. Recall that we have denoted the equivalence class $[\widehat{\alpha}^{-1}(s)]_{\sim_{\mathsf{V}}^A}$ simply by $[s]_{\sim_{\mathsf{V}}^A}$ for every $s \in M$.

The following corollary is analogous to Corollary 4.14 and it was also proven in [37, Lemma 4.10] for the case when the pseudovariety V is selfdual and $\mathsf{X} = \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. In this paper, we prove it easily using Corollary 5.16.

Corollary 5.17 *Let* V *be a locally finite pseudovariety. Let* $\alpha\colon A^* \to M$ *be a surjective* V*-compatible homomorphism into a finite monoid* M*. Let* $(e, \mathcal{E}) \in E\left(M \times 2^{M \times 2^M}\right)$ *be an* ω*-reducible pair for* $(\mathrm{Pol}\,\mathsf{V}, \mathsf{X})$*. Let*

$$\mathcal{S}_e = \left\{(s, S) \in M \times 2^M \mid (s, S)\ \omega\text{-reducible for } \mathsf{X},\ [e]_{\sim_{\mathsf{V}}^A} \leq [s]_{\sim_{\mathsf{V}}^A}\right\}.$$

Then the pair $(e, \mathcal{E}\mathcal{S}_e\mathcal{E})$ *is also* ω*-reducible for* $(\mathrm{Pol}\,\mathsf{V}, \mathsf{X})$.

Proof We will proceed analogously as in the proof of Corollary 4.14. Let $(e, \mathcal{E}) \in E\big(M \times 2^{M \times 2^M}\big)$ be an ω-reducible pair for $(\mathrm{Pol}\,\mathsf{V}, \mathsf{X})$. Then there exist an ω-word $x \in \Omega_A^\omega \mathsf{M}$ and a set $\mathcal{X} \subseteq \Omega_A^\omega \mathsf{M} \times 2^{\Omega_A^\omega \mathsf{M}}$ such that the following conditions are satisfied:

i) for every pair $(y, Y) \in \mathcal{X}$, the relations $\mathrm{Pol}\,\mathsf{V} \models x \leq y$, $\mathsf{X} \models y \leq Y$ hold,
ii) $\widehat{\alpha}(x) = e, \widehat{\alpha}(\mathcal{X}) = \mathcal{E}$.

Let $(s, S) \in \mathcal{S}_e$ be an arbitrary pair. By the definition of $\mathcal{S}_e$, the pair (s, S) is ω-reducible for X. This means that there exist an ω-word $\overline{y} \in \Omega_A^\omega \mathsf{M}$ and a set of ω-words $\overline{Y} \subseteq \Omega_A^\omega \mathsf{M}$ such that

I) $\mathsf{X} \models \overline{y} \leq \overline{Y}$,
II) $\widehat{\alpha}(\overline{y}) = s, \widehat{\alpha}(\overline{Y}) = S$.

Then, by the relation $[e]_{\sim_{\mathsf{V}}^A} \leq [s]_{\sim_{\mathsf{V}}^A}$, we have $\mathsf{V} \models x \leq \overline{y}$, i.e., $\mathsf{V}^{\mathrm{d}} \models \overline{y} \leq x$. Using Conditions I), II) and i), ii), we obtain that the the quadruple $(s, S, e, \mathcal{E})$ is ω-reducible for $(\mathsf{X}, \mathsf{V}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{V}, \mathsf{X})$. By Corollary 5.16, we obtain that the pair $(e, \mathcal{E}\mathcal{S}_e\mathcal{E})$ is ω-reducible for $(\mathrm{Pol}\,\mathsf{V}, \mathsf{X})$, as required. □

5.5.2 Employment of Lemma 5.9

Now we get to the announced application of Lemma 5.9. It is analogous to Lemma 4.15. Since we will use Corollary 5.17 in the proof, we need to assume again that the homomorphism $\alpha \colon A^* \to M$ is V-compatible.

As in Lemma 5.9, we will use the notation

$$N = 2^{|M| \cdot 2^{|M|}}.$$

Lemma 5.18 *Let* V *be a locally finite pseudovariety. Let* $\alpha \colon A^* \to M$ *be a surjective* V*-compatible homomorphism into a finite monoid* M*. Let* $n, k, l \in \mathbb{N}$, $k \geq N$, $l \geq N$*. Let* $u_1, \ldots, u_n \in A^*$ *be words such that* $\alpha(u_1) = \alpha(u_2) = \cdots = \alpha(u_n) = e$ *for some idempotent* $e \in E(M)$*. Let* $\mathcal{U}, \mathcal{U}_1, \ldots, \mathcal{U}_n \subseteq A^* \times 2^{A^*}$ *be sets satisfying the assumptions of Lemma 5.9. Let* $\mathcal{W}_1, \ldots, \mathcal{W}_m \subseteq A^* \times 2^{A^*}$*, where* $m \in \mathbb{N}$*, be sets satisfying Conditions 1 and 2 of Lemma 5.9. Let* U *be an arbitrary pseudovariety. If the pairs*

$$\big(e, \alpha(\mathcal{U}_1)\big), \ldots, \big(e, \alpha(\mathcal{U}_n)\big)$$

are ω*-reducible for* $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$*, and at least one of the following two conditions is satisfied:*

1. $\mathsf{U} = \mathsf{V}$ *or*
2. *for every set* $\mathcal{W}_i$ *of the form (ii) and every pair* $(\widetilde{v}, \widetilde{V}) \in \overline{\mathcal{U}} \cdot \mathcal{U}_{l'} \cdots \mathcal{U}_{k'}$*, the quadruple*

$$\big(\alpha(\widetilde{v}), \alpha(\widetilde{V}), e, \mathcal{E}\big)$$

is ω*-reducible for* $(\mathsf{X}, \mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U}, \mathsf{X})$*,*

then the pair $\big(e, \alpha(\mathcal{U})\big)$ *is also* ω*-reducible for* $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$.

Proof The proof is analogous to the proof of Lemma 4.15. By Condition 1 of Lemma 5.9, we know that $\mathcal{U} \subseteq \mathcal{W}_1 \cdots \mathcal{W}_m$. This means that

$$\big(e, \alpha(\mathcal{U})\big) \leq \big(e, \alpha(\mathcal{W}_1) \cdots \alpha(\mathcal{W}_m)\big) = \big(e, \alpha(\mathcal{W}_1)\big) \cdots \big(e, \alpha(\mathcal{W}_m)\big).$$

Hence, by Lemmas 5.12 and 5.13, it suffices to show that, for every $i \in \{1, \ldots, m\}$, the pair $\big(e, \alpha(\mathcal{W}_i)\big)$ is ω-reducible for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$.

Let $i \in \{1, \ldots, m\}$ be arbitrary. If the set $\mathcal{W}_i$ is of the form (i), we have

$$\big(e, \alpha(\mathcal{W}_i)\big) = \big(e, \alpha(\mathcal{U}_\iota)\big) \cdots \big(e, \alpha(\mathcal{U}_\kappa)\big).$$

Since the pairs $\big(e, \alpha(\mathcal{U}_\iota)\big), \ldots, \big(e, \alpha(\mathcal{U}_\kappa)\big)$ are ω-reducible for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$ by the assumption, we obtain that the pair $\big(e, \alpha(\mathcal{W}_i)\big)$ is also ω-reducible for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$, using Lemma 5.13.

Now suppose that the set $\mathcal{W}_i$ is of the form (ii). At first, since we have

$$\begin{aligned}(e, \mathcal{E}) &= \Big(e, \alpha\big((\mathcal{U}_{\kappa+1} \cdots \mathcal{U}_\lambda)^K\big)\Big) = \Big(e, \alpha(\mathcal{U}_{\kappa+1} \cdots \mathcal{U}_\lambda)\Big)^K = \\ &= \Big(\big(e, \alpha(\mathcal{U}_{\kappa+1})\big) \cdots \big(e, \alpha(\mathcal{U}_\lambda)\big)\Big)^K \end{aligned} \tag{18}$$

and the pairs $\big(e, \alpha(\mathcal{U}_{\kappa+1})\big), \ldots, \big(e, \alpha(\mathcal{U}_\lambda)\big)$ are ω-reducible for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$ by the assumption, we obtain that the pair $(e, \mathcal{E})$ is also ω-reducible for $(\mathrm{Pol}\,\mathsf{U}, \mathsf{X})$, using Lemma 5.13. We distinguish two cases:

1. $\mathsf{U} = \mathsf{V}$,
2. for every pair $(\widetilde{v}, \widetilde{V}) \in \overline{\mathcal{U}} \cdot \mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'}$, the quadruple

$$\big(\alpha(\widetilde{v}), \alpha(\widetilde{V}), e, \mathcal{E}\big)$$

is ω-reducible for $(\mathsf{X}, \mathsf{U}^{\mathrm{d}}, \mathrm{Pol}\,\mathsf{U}, \mathsf{X})$.

In the first case when $\mathsf{U} = \mathsf{V}$, we have that, for every pair $(\overline{v}, \overline{V}) \in \overline{\mathcal{U}}$, the relation

$$\mathrm{Pol}_{k-N}\,\mathsf{U} \models u_{\kappa+1} \ldots u_{\iota'-1} \leq \overline{v} \tag{19}$$

holds and the pair $(\overline{v}, \overline{V})$ is $(l-N)$-factorial for X by Lemma 5.9. Further, by the assumptions of Lemma 5.9, for every $i \in \{\iota', \ldots, \kappa'\}$ and every pair $(v_i, V_i) \in \mathcal{U}_i$, the relation

$$\mathrm{Pol}_{k-N}\,\mathsf{U} \models u_i \leq v_i \tag{20}$$

holds and the pair (v_i, V_i) is $(l-N)$-factorial for X. Multiplying the inequalities from (19) and (20), we obtain the relation

$$\mathrm{Pol}_{k-N}\,\mathsf{U} \models u_{\kappa+1} \ldots u_{\kappa'} = (u_{\kappa+1} \ldots u_{\iota'-1}) \cdot (u_{\iota'} \ldots u_{\kappa'}) \leq \overline{v} \cdot (v_{\iota'} \ldots v_{\kappa'}). \tag{21}$$

This implies that

$$\mathsf{U} \models u_{\kappa+1} \ldots u_{\kappa'} \leq \overline{v} \cdot (v_{\iota'} \ldots v_{\kappa'}).$$

Since $\alpha(u_{\kappa+1}\dots u_{\kappa'}) = e$, we obtain that $[e]_{\sim^A_U} \leq [\overline{v}\cdot(v_{\iota'}\dots v_{\kappa'})]_{\sim^A_U}$. Further, recall that the pairs $(\overline{v},\overline{V})$, (v_i, V_i), $i=\iota',\dots\kappa'$, are $(l-N)$-factorial for X. This implies that the pairs $\big(\alpha(\overline{v}),\alpha(\overline{V})\big)$, $\big(\alpha(v_i),\alpha(V_i)\big)$, $i=\iota',\dots\kappa'$, are ω-reducible for X.

Let $(\widetilde{v},\widetilde{V}) \in \overline{\mathcal{U}}\cdot\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'}$ be an arbitrary pair. Then, by Proposition 5.4, there exist pairs $(\overline{v},\overline{V})\in\overline{\mathcal{U}}$, $(v_i,V_i)\in\mathcal{U}_i$ for $i=\iota',\dots,\kappa'$ such that $\widetilde{v}=\overline{v}\cdot v_{\iota'}\dots v_{\kappa'}$ and $\widetilde{V}\subseteq\overline{V}\cdot V_{\iota'}\cdots V_{\kappa'}$. Then we obtain that the pair

$$\begin{aligned}\big(\alpha(\widetilde{v}),\alpha(\widetilde{V})\big) &\leq \big(\alpha(\overline{v}\cdot v_{\iota'}\dots v_{\kappa'}),\alpha(\overline{V}\cdot V_{\iota'}\cdots V_{\kappa'})\big) = \\ &= \big(\alpha(\overline{v}),\alpha(\overline{V})\big)\cdot\big(\alpha(v_{\iota'}),\alpha(V_{\iota'})\big)\cdots\big(\alpha(v_{\kappa'}),\alpha(V_{\kappa'})\big)\end{aligned}$$

is ω-reducible for X, using Lemmas 5.12 and 5.13.

Altogether, we have

$$\big(\alpha(\widetilde{v}),\alpha(\widetilde{V})\big)\in\mathcal{S}_e := \big\{(s,S)\in M\times 2^M\ \omega\text{-reducible for X} \mid [e]_{\sim^A_U}\leq[s]_{\sim^A_U}\big\}. \quad (22)$$

Since this holds for every pair $(\widetilde{v},\widetilde{V})\in\overline{\mathcal{U}}\cdot\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'}$, we obtain

$$\alpha(\overline{\mathcal{U}}\cdot\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'})\subseteq\mathcal{S}_e.$$

Then, by Corollary 5.17 and Lemma 5.12, the pair

$$\big(e,\mathcal{E}\cdot\alpha(\overline{\mathcal{U}}\cdot\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'})\cdot\mathcal{E}\big)\leq(e,\mathcal{E}\mathcal{S}_e\mathcal{E})$$

is ω-reducible for (Pol U, X).

In the second case, for every pair $(\widetilde{v},\widetilde{V})\in\overline{\mathcal{U}}\cdot\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'}$, the quadruple

$$\big(\alpha(\widetilde{v}),\alpha(\widetilde{V}),e,\mathcal{E}\big)$$

is ω-reducible for (X, U^{d}, Pol U, X), by the assumption. Then, by Corollary 5.16, we obtain that the pair

$$\Big(e,\mathcal{E}\cdot\alpha(\overline{\mathcal{U}}\cdot\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'})\cdot\mathcal{E}\Big)$$

is ω-reducible for (Pol U, X).

Finally, in both cases, we have

$$\begin{aligned}\alpha(\mathcal{W}_i) &= \alpha(\mathcal{U}_\iota\cdots\mathcal{U}_\kappa)\cdot\alpha(\overline{\mathcal{U}})\cdot\alpha(\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'}) = \\ &= \alpha(\mathcal{U}_\iota\cdots\mathcal{U}_\kappa)\cdot\mathcal{E}\cdot\alpha(\overline{\mathcal{U}})\cdot\alpha(\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'})\cdot\mathcal{E} = \\ &= \alpha(\mathcal{U}_\iota)\cdots\alpha(\mathcal{U}_\kappa)\cdot\mathcal{E}\cdot\alpha(\overline{\mathcal{U}}\cdot\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'})\cdot\mathcal{E}.\end{aligned}$$

Then the pair

$$\big(e,\alpha(\mathcal{W}_i)\big)=\big(e,\alpha(\mathcal{U}_\iota)\big)\cdots\big(e,\alpha(\mathcal{U}_\kappa)\big)\cdot\big(e,\mathcal{E}\cdot\alpha(\overline{\mathcal{U}}\cdot\mathcal{U}_{\iota'}\cdots\mathcal{U}_{\kappa'})\cdot\mathcal{E}\big)$$

is ω-reducible for (Pol U, X) by Lemma 5.13. We have finished the proof of Lemma 5.18. □

Remark 5.19

1. In the case when $\mathsf{U} = \mathsf{V}$, one can prove that Condition 2 of Lemma 5.18 is also satisfied. The procedure is similar to the proof of Corollary 5.17. It suffices to use
   - Relation (22), which implies that, for every pair $(\widetilde{v}, \widetilde{V}) \in \overline{\mathcal{U}} \cdot \mathcal{U}_{\iota'} \cdots \mathcal{U}_{\kappa'}$ and every ω-word $x \in \widehat{\alpha}^{-1}(e)$, we have $\mathsf{U}^{\mathrm{d}} \models \widetilde{v} \leq x$, $\mathsf{X} \models \widetilde{v} \leq \widetilde{V}$,
   - the ω-reducibility of the pair (e, E), following from Relation (18), and
   - Corollary 5.16.
2. In the case when $\mathsf{U} \neq \mathsf{V}$, the assumption of V-compatibility of α is superfluous.
3. The case $\mathsf{U} \neq \mathsf{V}$ is not used further in this paper. It is stated in Lemma 5.18 to see how a further generalization of our approach to higher half levels of concatenation hierarchies could look like.

5.6 Main proof

Now we proceed to prove the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$. The procedure of the proof is similar to the procedure in the author's PhD thesis [36, Subsection 4.2.4]. The presentation differs in part due to the employment of k-factorial pairs and Lemmas 4.15 and 5.18 in this paper. The proof is similar also to the proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}\,\mathsf{V}}[M]$ from Subsection 4.4, although it is considerably more complicated – we will need to use both Lemma 4.15 on the ω-reducibility of pairs $(s, S) \in M \times 2^M$ for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ and Lemma 5.18 on the ω-reducibility of pairs $(s, \mathcal{S}) \in M \times 2^{M \times 2^M}$ for $\big(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$.

In the proof, we use the "double stratification" of the pseudovariety $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ from Subsection 3.3.3. We choose appropriate indices ν_0 and μ_0 for which

$$\mathcal{P}_{\mathrm{Pol}_{\mu_0}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}}}[M] = \mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}}}[M] = \mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M] \tag{23}$$

(such indices must exist by Proposition 3.7). The whole proof is done using the locally finite pseudovarieties $\mathrm{Pol}_\mu(\mathrm{Pol}_\nu\mathsf{V})^{\mathrm{d}}$ for $\nu \leq \nu_0$, $\mu \leq \mu_0$. However, we do not see that Relation (23) holds until finishing the proof.

Let V be a *locally finite* pseudovariety of ordered monoids, A be an alphabet, and M be a finite monoid. Recall that we worked with a surjective V-*compatible* homomorphism $\alpha\colon A^* \to M$. However, for the proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$, we need to use a surjective $(\mathrm{Pol}_1\mathsf{V})$-*compatible* homomorphism $\alpha\colon A^* \to M$. Thus, let α be such a homomorphism.

Let

$$\nu_0 := 1 + 9|M| \cdot 2^{|M| \cdot 2^{|M|}}.$$

Let $\alpha'\colon A^* \to M \times \overline{\Omega}_A(\mathrm{Pol}_{\nu_0}\mathsf{V})$ be the $(\mathrm{Pol}_{\nu_0}\mathsf{V})$-completion of the homomorphism α. Then we have

$$\forall u \in A^*\colon \quad \alpha'(u) = \big(\alpha(u), [u]_{\sim_{\mathrm{Pol}_{\nu_0}\mathsf{V}}}\big).$$

Denote $M' := M_{\alpha'} = \alpha'(A^*)$. Further, analogously to the case of the homomorphism α, we denote the equivalence class $[(\widehat{\alpha'})^{-1}(s)]_{\sim_{\mathrm{Pol}_{\nu_0}\mathsf{V}}}$ simply by $[s]_{\sim_{\mathrm{Pol}_{\nu_0}\mathsf{V}}}$ for every $s \in M'$.

Let

$$\xi := 2 + 9|M| \cdot 2^{|M| \cdot 2^{|M|}} + 2^{2|M|} = \nu_0 + 1 + 2^{2|M|}$$

and

$$\mu_0 := 1 + \left(3 \cdot 2^{|M|} + \xi\right) \cdot 3|M'|.$$

Theorem 5.20 *Let* V *be a locally finite pseudovariety of ordered monoids and let* M *be a finite monoid. Then the set* $\mathcal{P}_{\mathrm{Pol(Pol\,V)^d}}[M]$ *is* ω*-reducible.*

Moreover, if there is a surjective $(\mathrm{Pol}_1\mathsf{V})$*-compatible homomorphism* $\alpha\colon A^* \to M$ *and* ν_0, μ_0 *are the constants described above, then the equalities*

$$\mathcal{P}_{\mathrm{Pol(Pol\,V)^d}}[M] = \mathcal{P}_{\mathrm{Pol(Pol_{\nu_0}V)^d}}[M] = \mathcal{P}_{\mathrm{Pol_{\mu_0}(Pol_{\nu_0}V)^d}}[M]$$

hold.

Proof By Lemma 3.1, to show that the set $\mathcal{P}_{\mathrm{Pol(Pol\,V)^d}}[M]$ is ω-reducible, it suffices to consider the case when we have a surjective $(\mathrm{Pol}_1\mathsf{V})$-compatible homomorphism $\alpha\colon A^* \to M$.

To prove the whole theorem, we use Proposition 3.4. By Propositions 3.14 and 3.15, $\mathrm{Pol}_{\mu_0}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}}$ is a locally finite pseudovariety such that

$$\mathrm{Pol}_{\mu_0}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}} \subseteq \mathrm{Pol}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}} \subseteq \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}.$$

This implies, by the definition of a set of the form $\mathcal{P}_{\mathsf{V}}[M]$, that

$$\mathcal{P}_{\mathrm{Pol(Pol\,V)^d}}[M] \subseteq \mathcal{P}_{\mathrm{Pol(Pol_{\nu_0}V)^d}}[M] \subseteq \mathcal{P}_{\mathrm{Pol_{\mu_0}(Pol_{\nu_0}V)^d}}[M]. \qquad (24)$$

We show that the assumptions of Proposition 3.4 for the locally finite pseudovariety $\mathrm{Pol}_{\mu_0}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}}$, seen as a subpseudovariety of $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, are satisfied. This will imply, by Proposition 3.4 and Relation (24), that the statement of Theorem 5.20 holds.

Let $u \in A^*$ be an arbitrary word and $U \subseteq A^*$ be an arbitrary set of words satisfying the relation $\mathrm{Pol}_{\mu_0}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}} \models u \le U$. We need to show that there exist an ω-word $x \in \Omega_A^{\omega}\mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^{\omega}\mathsf{M}$ satisfying the following conditions:

i) $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}} \models x \le X$,
ii) $\widehat{\alpha}(x) = \alpha(u), \widehat{\alpha}(X) = \alpha(U)$.

We prove this by the induction on the height of the word u in a fixed factorization forest d' for α'. More precisely, we prove the following theorem.

Let d' be a factorization forest for α'. For every $u \in A^*$, denote

$$\mu(u) := 1 + \left(3 \cdot 2^{|M|} + \xi\right) \cdot h_{d'}(u).$$

Further, let

$$\mathsf{W} := (\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}}.$$

Theorem 5.21 *Let d' be a factorization forest for α'. Let $u \in A^*$, $U \subseteq A^*$. Let γ be an infix of u. Suppose that*

$$\mathrm{Pol}_{\mu(u)}\mathsf{W} \models \gamma \leq U.$$

Then the pair $(\alpha(\gamma), \alpha(U))$ *is ω-reducible for* $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

Note that this theorem is analogous to Theorem 4.18, but a bit more general – we need to consider not only a given word u, but all infixes γ of u. We will see the reason for this generalization later in the proof.

At first, let us show how to use Theorem 5.21 to complete the proof of Theorem 5.20. Choose a factorization forest d' for α of height at most $3|M'|$, which exists by Theorem 2.4. Then

$$\mu(u) = 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot h_{d'}(u) \leq 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot 3|M'| = \mu_0.$$

This implies that the relation $\mathrm{Pol}_{\mu(u)}\mathsf{W} \models u \leq U$ holds. By Theorem 5.21, we obtain that the pair $(\alpha(u), \alpha(U))$ is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. This means that there exist an ω-word $x \in \Omega_A^{\omega}\mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^{\omega}\mathsf{M}$ satisfying Conditions i) and ii). We have proven that the assumptions of Theorem 3.4 are satisfied. Hence, by Proposition 3.4 and Relation (24), we obtain that

$$\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M] = \mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}}}[M] = \mathcal{P}_{\mathrm{Pol}_{\mu_0}(\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}}}[M]$$

and the set $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ is ω-reducible. □

It remains to prove Theorem 5.21.

Proof of Theorem 5.21 The proof goes by induction on $h_{d'}(u)$. If $h_{d'}(u) = 0$, then $u \in A \cup \{\varepsilon\}$ and $\mu(u) = 1$. Hence $\gamma \in A \cup \{\varepsilon\}$ and $\mathrm{Pol}_1\mathsf{W} \models \gamma \leq U$. We have[9]

$$\{\varepsilon\} = \bigcap_{a \in A} (A^*aA^*)^{\mathrm{C}} = \Big(\bigcup_{a \in A} A^*aA^*\Big)^{\mathrm{C}} \in \text{Co-}Pol_1(\mathcal{V})(A) = Pol_0\big(\text{Co-}Pol_1(\mathcal{V})\big)(A)$$

and, for every $a \in A$,

$$\begin{aligned}\{a\} &= A^*aA^* \cap \bigcap_{b \in A, b \neq a} (A^*bA^*)^{\mathrm{C}} \cap (A^*aA^*aA^*)^{\mathrm{C}} = \\ &= A^*aA^* \cap \Big(\bigcup_{b \in A, b \neq a} A^*bA^* \cup A^*aA^*aA^*\Big)^{\mathrm{C}} \in Pol_1\big(\text{Co-}Pol_2(\mathcal{V})\big)(A).\end{aligned}$$

Since we have $\mathsf{W} = (\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}}$ and $\nu_0 > 2$, the property $\mathrm{Pol}_1\mathsf{W} \models \gamma \leq U$ implies that $U \subseteq \{\gamma\}$. Then the pair $(\alpha(\gamma), \alpha(U)) \leq (\alpha(\gamma), \{\alpha(\gamma)\})$ is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ by Lemmas 4.8 and 4.9.

Now suppose that $h_{d'}(u) \geq 1$. Let $d'(u) = (u_1, \ldots, u_n)$. Then two cases can occur:

[9] In the following relations, we denote the complement of a language $L \subseteq A^*$ by L^{C}.

a) $n = 2$,
b) $n > 2$.

The proceeding in Case a) is straightforward. Let $d'(u) = (u_1, u_2)$. We have $u = u_1 u_2$. Then there exist $\gamma_1, \gamma_2 \in A^*$ such that $\gamma = \gamma_1\gamma_2$, γ_1 is an infix of u_1, γ_2 is an infix of u_2. For $i = 1, 2$, denote by U_i the maximal subset of A^* satisfying

$$\mathrm{Pol}_{\mu(u_i)}\mathsf{W} \models \gamma_i \le U_i.$$

Then, by the induction assumption, the pairs $\big(\alpha(\gamma_1), \alpha(U_1)\big)$, $\big(\alpha(\gamma_2), \alpha(U_2)\big)$ are ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. We have

$$\begin{aligned} \mu(u_i) &= 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot h_{d'}(u_i) \le \\ &\le 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot (h_{d'}(u) - 1) = \\ &= 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot h_{d'}(u) - \big(3 \cdot 2^{|M|} + \xi\big) = \\ &= \mu(u) - \big(3 \cdot 2^{|M|} + \xi\big) < \\ &< \mu(u) - 1. \end{aligned} \tag{25}$$

Then, by Lemma 4.4, we obtain that $U \subseteq U_1 U_2$. This implies that

$$\big(\alpha(\gamma), \alpha(U)\big) \le \big(\alpha(\gamma_1), \alpha(U_1)\big) \cdot \big(\alpha(\gamma_2), \alpha(U_2)\big).$$

Using Lemmas 4.9 and 4.10, we obtain that the pair $\big(\alpha(\gamma), \alpha(U)\big)$ is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

Now suppose that $h_{d'}(u) \ge 1$ and $d'(u) = (u_1, \ldots, u_n)$, where $n > 2$. Then there exists an idempotent $\epsilon = (e, [e]_{\sim_{\mathrm{Pol}_{\nu_0}\mathsf{V}}}) \in E(M') \subseteq E(M) \times E(A^*/{\sim_{\mathrm{Pol}_{\nu_0}\mathsf{V}}})$ such that $\alpha'(u_1) = \cdots = \alpha'(u_n) = \alpha'(u) = \epsilon$. Further, we have $u = u_1 \ldots u_n$. Hence there exist indices i_1, i_2, $1 \le i_1 \le i_2 \le n$ and words $\gamma_{i_1}, \ldots, \gamma_{i_2} \in A^*$ such that $\gamma = \gamma_{i_1} \ldots \gamma_{i_2}$, γ_{i_1} is an infix of u_{i_1}, γ_{i_2} is an infix of u_{i_2}, $\gamma_i = u_i$ for $i = i_1 + 1, \ldots, i_2 - 1$. Denote

$$U_{i_1} = \{w \in A^* \mid \mathrm{Pol}_{\mu(u_{i_1})}\mathsf{W} \models \gamma_{i_1} \le w\},$$

$$U_{i_2} = \{w \in A^* \mid \mathrm{Pol}_{\mu(u_{i_2})}\mathsf{W} \models \gamma_{i_2} \le w\}.$$

Then, by the induction assumption, the pairs $\big(\alpha(\gamma_{i_1}), \alpha(U_{i_1})\big)$, $\big(\alpha(\gamma_{i_2}), \alpha(U_{i_2})\big)$ are ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

If $i_2 = i_1$, then we have $\gamma = \gamma_{i_1} \ldots \gamma_{i_2} = \gamma_{i_1}$. Using the assumption

$$\mathrm{Pol}_{\mu(u)}\mathsf{W} \models \gamma_{i_1} = \gamma \le U$$

and the relation $\mu(u_{i_1}) < \mu(u)$ (following from the relation $h_{d'}(u_{i_1}) < h_{d'}(u)$), we obtain that $U \subseteq U_{i_1}$. Then the pair

$$\big(\alpha(\gamma), \alpha(U)\big) \le \big(\alpha(\gamma_{i_1}), \alpha(U_{i_1})\big)$$

is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ by Lemma 4.9.

If $i_2 = i_1 + 1$, then we have $\gamma = \gamma_{i_1} \dots \gamma_{i_2} = \gamma_{i_1}\gamma_{i_2}$, where γ_{i_1} is an infix of u_{i_1} and γ_{i_2} is an infix of u_{i_2}. We can proceed in the same way as in Case a) (when $d'(u) = (u_1, u_2)$).

It remains to consider the case when $i_2 > i_1 + 1$. Let

$$V = \{w \in A^* \mid \mathrm{Pol}_{\mu(u)-2}\mathsf{W} \models \gamma_{i_1+1} \dots \gamma_{i_2-1} \leq w\}.$$

Notice that we have $\mu(u_{i_1}) < \mu(u) - 2$ and $\mu(u_{i_2}) < \mu(u) - 2$ analogously to (25). Hence we obtain that $U \subseteq U_{i_1} V U_{i_2}$ by Lemma 4.4. The rest of the proof is devoted to proving that the pair

$$\big(\alpha(\gamma_{i_1+1} \dots \gamma_{i_2-1}), \alpha(V)\big) = \big(\alpha(u_{i_1+1} \dots u_{i_2-1}), \alpha(V)\big) = \big(e, \alpha(V)\big)$$

is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. Then we will obtain that the pair

$$\big(\alpha(\gamma), \alpha(U)\big) \leq \big(\alpha(\gamma_{i_1}), \alpha(U_{i_1})\big) \cdot \big(e, \alpha(V)\big) \cdot \big(\alpha(\gamma_{i_2}), \alpha(U_{i_2})\big) \tag{26}$$

is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ by Lemmas 4.9 and 4.10.

To prove the ω-reducibility of the pair $\big(e, \alpha(V)\big)$ for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, we use Lemma 4.15. Let

$$k := \mu(u) - 2 - 2 \cdot 2^{|M|}.$$

Then we have

$$\begin{aligned} k &= 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot h_{d'}(u) - 2 - 2 \cdot 2^{|M|} \geq \\ &\geq 1 + 3 \cdot 2^{|M|} + \xi - 2 - 2 \cdot 2^{|M|} = \\ &= 1 + 2^{|M|} + \xi - 2 = 2^{|M|} + \xi - 1 = \\ &= 2^{|M|} + (2 + 9|M| \cdot 2^{|M| \cdot 2^{|M|}} + 2^{2|M|}) - 1 > \\ &> 2^{|M| \cdot 2^{|M|}} \end{aligned}$$

and

$$\begin{aligned} k + 2 \cdot \mathrm{i}_{k-2^{|M|}}(u_{i_1+1}, \dots, u_{i_2-1}) &= \mu(u) - 2 - 2 \cdot 2^{|M|} \\ &\quad + 2 \cdot \mathrm{i}_{k-2^{|M|}}(u_{i_1+1}, \dots, u_{i_2-1}) \leq \\ &\leq \mu(u) - 2 - 2 \cdot 2^{|M|} + 2 \cdot 2^{|M|} = \\ &= \mu(u) - 2. \end{aligned}$$

By the definition of the set V, we have

$$\mathrm{Pol}_{\mu(u)-2}\mathsf{W} \models u_{i_1+1} \dots u_{i_2-1} \leq V.$$

By the preceding, this implies that

$$\mathrm{Pol}_{k+2\cdot \mathrm{i}_{k-2^{|M|}}(u_{i_1+1},\ldots,u_{i_2-1})}\mathsf{W} \models u_{i_1+1}\ldots u_{i_2-1} \leq V.$$

Further, for $i = i_1 + 1, \ldots, i_2 - 1$, let U_i be the maximal subset of A^* satisfying

$$\mathrm{Pol}_{k-2^{|M|}}\,\mathsf{W} \models u_i \leq U_i.$$

We have

$$\begin{aligned}\mu(u_i) &= 1 + \big(3\cdot 2^{|M|} + \xi\big)\cdot h_{d'}(u_i) \leq \\ &\leq 1 + \big(3\cdot 2^{|M|} + \xi\big)\cdot \big(h_{d'}(u) - 1\big) = \\ &= 1 + \big(3\cdot 2^{|M|} + \xi\big)\cdot h_{d'}(u) - \big(3\cdot 2^{|M|} + \xi\big) = \\ &= \mu(u) - 3\cdot 2^{|M|} - \xi\end{aligned}$$

and

$$k - 2^{|M|} = \mu(u) - 2 - 2\cdot 2^{|M|} - 2^{|M|} = \mu(u) - 2 - 3\cdot 2^{|M|} > \mu(u) - 3\cdot 2^{|M|} - \xi.$$

Altogether, we obtain

$$\mu(u_i) < k - 2^{|M|}.$$

This implies that

$$\mathrm{Pol}_{\mu(u_i)}\mathsf{W} \models u_i \leq U_i \quad \text{for } i = i_1 + 1, \ldots, i_2 - 1.$$

Then, by the induction assumption, the pairs

$$\big(\alpha(u_{i_1+1}), \alpha(U_{i_1+1})\big), \ldots, \big(\alpha(u_{i_2-1}), \alpha(U_{i_2-1})\big)$$

are ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

Let $W_1, \ldots, W_m \subseteq A^*$ ($m \in \mathbb{N}$) be sets satisfying Conditions 1 and 2 of Lemma 4.5. Let W_i, $i \in \{1, \ldots, m\}$, be an arbitrary set of the form (ii), i.e.,

$$W_i = U_\iota \cdot U_{\iota+1} \cdots U_\kappa \cdot \overline{U} \cdot U_{\iota'} \cdot U_{\iota'+1} \cdots U_{\kappa'}$$

for some set $\overline{U} \subseteq A^*$ such that

$$\mathrm{Pol}_{k-2^{|M|}}\,\mathsf{W} \models u_{\kappa+1}\ldots u_{\iota'-1} \leq \overline{U},$$

for some indices $\iota, \kappa, \iota', \kappa' \in \{i_1 + 1, \ldots, i_2 - 1\}$, $\iota \leq \kappa < \iota' \leq \kappa'$ for which there exists an idempotent $E \in E(2^M)$ satisfying the relations

$$\alpha(U_\iota \cdots U_\kappa) = \alpha(U_\iota \cdots U_\kappa)\cdot E, \quad \alpha(U_{\iota'} \cdots U_{\kappa'}) = \alpha(U_{\iota'} \cdots U_{\kappa'})\cdot E,$$

and

$$E = \big(\alpha(U_{\kappa+1} \cdots U_\lambda)\big)^K$$

for some index $\lambda \in \{\kappa+1, \ldots, i_2-1\}$, $\lambda-\kappa \leq 2^{|M|}$, and a number $K \in \{1, \ldots, 2^{|M|}\}$
.

To complete the proof of the ω-reducibility of the pair $\big(e, \alpha(V)\big)$ for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ by Lemma 4.15, we need to show that, for every word $\widetilde{v} \in \overline{U} \cdot U_{\iota'} \cdots U_{\kappa'}$, the triple

$$\big(\alpha(\widetilde{v}), e, E\big)$$

is ω-reducible for $\big(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$. In fact, we need to prove a bit more general statement, stated in Lemma 5.22 below.

Let d be a factorization forest for α. For every word $p \in A^*$, denote

$$\nu(p) := 1 + 3 \cdot 2^{|M| \cdot 2^{|M|}} \cdot h_d(p)$$

and

$$\lambda(p) := 3 \cdot 2^{|M| \cdot 2^{|M|}} \cdot h_d(p) = \nu(p) - 1.$$

Lemma 5.22 *Let d be a factorization forest for α. Let $p \in A^*$, $\mathcal{P} \subseteq A^* \times 2^{A^*}$ be such that, for every $(q, Q) \in \mathcal{P}$, the relation $\mathrm{Pol}_{\nu(p)}\,\mathsf{V} \models p \leq q$ holds and the pair (q, Q) is $\lambda(p)$-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. Then the pair $\big(\alpha(p), \alpha(\mathcal{P})\big)$ is ω-reducible for $\big(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$.*

To be able to use Lemma 5.22, we need also the following lemma.

Let

$$C := k - 2^{|M|} - 9|M| \cdot 2^{|M| \cdot 2^{|M|}}.$$

Lemma 5.23 *Let $\chi \in \mathbb{N}_0$ be an arbitrary number. Let $q \in A^*$ be an infix of $(u_{\kappa+1} \ldots u_\lambda)^K$ and $Q \subseteq A^*$ be a set such that the relation*

$$\mathrm{Pol}_{C+\chi}\,\mathsf{W} \models q \leq Q$$

holds. Then the pair (q, Q) is χ-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

Notice that the statement of Lemma 5.23 uses the word $(u_{\kappa+1} \ldots u_\lambda)^K$ and the number k, which come from the proof of Theorem 5.21. To prove Lemma 5.23, we need to use the induction assumption of Theorem 5.21.

On the other hand, both the statement and the proof of Lemma 5.22 are independent of Theorem 5.21.

Let us now explain how to use Lemmas 5.22 and 5.23 to complete the proof of Theorem 5.21. Since we have

$$\mathrm{Pol}_{k-2^{|M|}}\mathsf{W} \models u_{\kappa+1} \ldots u_{\iota'-1} \leq \overline{U}$$

by the assumption and

$$\mathrm{Pol}_{k-2^{|M|}}\mathsf{W} \models u_i \leq U_i \quad \text{for } i = \iota', \ldots, \kappa'$$

by the definition, we obtain

$$\mathrm{Pol}_{k-2^{|M|}}\mathsf{W} \models u_{\kappa+1}\dots u_{\kappa'} \leq \overline{U}\cdot U_{\iota'}\cdots U_{\kappa'}.$$

Let $\widetilde{v} \in \overline{U}\cdot U_{\iota'}\cdots U_{\kappa'}$ be an arbitrary word. Since we have

$$\mathrm{Pol}_{k-2^{|M|}}\mathsf{W} \models u_{\kappa+1}\dots u_{\kappa'} \leq \widetilde{v} \quad \text{and} \quad \mathsf{W} = (\mathrm{Pol}_{\nu_0}\mathsf{V})^{\mathrm{d}},$$

we obtain that

$$\mathrm{Pol}_{\nu_0}\mathsf{V} \models \widetilde{v} \leq u_{\kappa+1}\dots u_{\kappa'}.$$

Furthermore, the equality $\alpha'\big((u_{\kappa+1}\dots u_\lambda)^K\big) = \epsilon = \alpha'(u_{\kappa+1}\dots u_{\kappa'})$ implies that

$$\mathrm{Pol}_{\nu_0}\mathsf{V} \models (u_{\kappa+1}\dots u_\lambda)^K = u_{\kappa+1}\dots u_{\kappa'}.$$

Altogether, we obtain

$$\mathrm{Pol}_{\nu_0}\mathsf{V} \models \widetilde{v} \leq (u_{\kappa+1}\dots u_\lambda)^K.$$

Choose a factorization forest d for α of height at most $3|M|$, which exists by Theorem 2.4. Then we have

$$\nu(\widetilde{v}) = 1 + 3\cdot 2^{|M|\cdot 2^{|M|}}\cdot h_d(\widetilde{v}) \leq 1 + 9|M|\cdot 2^{|M|\cdot 2^{|M|}} = \nu_0.$$

This implies that

$$\mathrm{Pol}_{\nu(\widetilde{v})}\mathsf{V} \models \widetilde{v} \leq (u_{\kappa+1}\dots u_\lambda)^K. \tag{27}$$

Further, recall that we have $\mathrm{Pol}_{k-2^{|M|}}\mathsf{W} \models u_i \leq U_i$ for every i by the definition of U_i. This implies that

$$\mathrm{Pol}_{k-2^{|M|}}\mathsf{W} \models (u_{\kappa+1}\dots u_\lambda)^K \leq (U_{\kappa+1}\dots U_\lambda)^K.$$

Then, by Lemma 5.23, we obtain that the pair $\big((u_{\kappa+1}\dots u_\lambda)^K, (U_{\kappa+1}\dots U_\lambda)^K\big)$ is $\big(9|M|\cdot 2^{|M|\cdot 2^{|M|}}\big)$-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. Since

$$\lambda(p) = 3\cdot 2^{|M|\cdot 2^{|M|}}\cdot h_d(p) \leq 9|M|\cdot 2^{|M|\cdot 2^{|M|}},$$

this implies that the pair $\big((u_{\kappa+1}\dots u_\lambda)^K, (U_{\kappa+1}\dots U_\lambda)^K\big)$ is $\lambda(p)$-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ by Lemma 5.2. Then, using Relation (27) and Lemma 5.22, we obtain that the pair

$$\Big(\alpha(\widetilde{v}), \Big\{\Big(\alpha\left((u_{\kappa+1}\dots u_\lambda)^K\right), \alpha\left((U_{\kappa+1}\dots U_\lambda)^K\right)\Big)\Big\}\Big) = \big(\alpha(\widetilde{v}), \{(e, E)\}\big) \in M \times 2^{M\times 2^M}$$

is ω-reducible for $\big(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$, i.e., the triple $\big(\alpha(\widetilde{v}), e, E\big)$ is ω-reducible for $\big(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$.

Then, by Lemma 4.15, the pair $\big(e, \alpha(V)\big)$ is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. By Relation (26) and Lemmas 4.9, 4.10, we obtain that the pair $\big(\alpha(\gamma), \alpha(U)\big)$ is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, as required. □

Note that, to prove Theorem 5.21, it sufficed to use Lemma 5.23 just for the case when $q = (u_{\kappa+1} \dots u_\lambda)^K$. However, to prove the statement of Lemma 5.23 for $q = (u_{\kappa+1} \dots u_\lambda)^K$, we need to consider an arbitrary infix of $(u_{\kappa+1} \dots u_\lambda)^K$ – as we will see below.

It remains to prove Lemmas 5.22 and 5.23. We begin with the proof of Lemma 5.23, which uses the induction assumption of Theorem 5.21.

Proof of Lemma 5.23 We prove the lemma by induction on χ. At first, suppose that $\chi = 0$. We have

$$\mathrm{Pol}_C\, \mathsf{W} \models q \leq Q$$

by the assumption. We need to prove that the pair (q, Q) is 0-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, i.e., that the pair $\big(\alpha(q), \alpha(Q)\big)$ is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. By the assumption, q is an infix of $(u_{\kappa+1} \dots u_\lambda)^K$. Hence there exist words

$$q^1_{\kappa+1}, \dots, q^1_\lambda, q^2_{\kappa+1}, \dots q^2_\lambda, \dots, q^K_{\kappa+1}, \dots q^K_\lambda \in A^*$$

such that

$$q = (q^1_{\kappa+1} \dots q^1_\lambda)(q^2_{\kappa+1} \dots q^2_\lambda) \dots (q^K_{\kappa+1} \dots q^K_\lambda) \tag{28}$$

and, for every $i \in \{\kappa+1, \dots, \lambda\}$ and every $j \in \{1, \dots, K\}$, q^j_i is an infix of u_i. For every $i \in \{\kappa+1, \dots, \lambda\}$ and every $j \in \{1, \dots, K\}$, let $Q^j_i \subseteq A^*$ be the maximal set satisfying

$$\mathrm{Pol}_{C-(\lambda-\kappa)\cdot K}\, \mathsf{W} \models q^j_i \leq Q^j_i.$$

By Lemma 4.4, we obtain that

$$Q \subseteq (Q^1_{\kappa+1} \cdots Q^1_\lambda) \cdot (Q^2_{\kappa+1} \cdots Q^2_\lambda) \cdots (Q^K_{\kappa+1} \cdots Q^K_\lambda). \tag{29}$$

For every $i \in \{\kappa+1, \dots, \lambda\}$, we have

$$\begin{aligned}
\mu(u_i) &= 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot h_{d'}(u_i) \leq \\
&\leq 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot \big(h_{d'}(u) - 1\big) = \\
&= 1 + \big(3 \cdot 2^{|M|} + \xi\big) \cdot h_{d'}(u) - \big(3 \cdot 2^{|M|} + \xi\big) = \\
&= \mu(u) - 3 \cdot 2^{|M|} - \xi = \\
&= \mu(u) - 3 \cdot 2^{|M|} - 2 - 9|M| \cdot 2^{|M| \cdot 2^{|M|}} - 2^{2|M|} = \\
&= k - 2^{|M|} - 9|M| \cdot 2^{|M| \cdot 2^{|M|}} - 2^{2|M|} = \\
&= C - 2^{2|M|}.
\end{aligned}$$

Further, we have $\lambda - \kappa \leq 2^{|M|}$ and $K \leq 2^{|M|}$. This implies that

$$(\lambda - \kappa) \cdot K \leq 2^{2|M|}.$$

Then we obtain

$$\mu(u_i) \leq C - (\lambda - \kappa) \cdot K.$$

This implies that, for every $i \in \{\kappa + 1, \ldots, \lambda\}$ and every $j \in \{1, \ldots, K\}$, the relation

$$\mathrm{Pol}_{\mu(u_i)}\mathsf{W} \models q_i^j \leq Q_i^j$$

holds. Then, by the induction assumption *(Theorem 5.21)*, we obtain that, for every $i \in \{\kappa + 1, \ldots, \lambda\}$ and every $j \in \{1, \ldots, K\}$, the pair $\big(\alpha(q_i^j), \alpha(Q_i^j)\big)$ is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. Finally, using Relations (28), (29) and Lemmas 4.9 and 4.10, we obtain that the pair $\big(\alpha(q), \alpha(Q)\big)$ is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, i.e., the pair (q, Q) is 0-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

Now suppose that $\chi \geq 1$. By the assumption, we have the relation

$$\mathrm{Pol}_{C+\chi}\mathsf{W} \models q \leq Q. \tag{30}$$

We need to prove that the pair (q, Q) is χ-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. Let $q', q'' \in A^*$ be arbitrary words such that $q = q'q''$. By Relation (30) and by Lemma 4.4, there exist sets $Q', Q'' \subseteq A^*$ such that

- $Q \subseteq Q'Q''$,
- the relations

$$\mathrm{Pol}_{C+\chi-1}\mathsf{W} \models q' \leq Q'$$

  and

$$\mathrm{Pol}_{C+\chi-1}\mathsf{W} \models q'' \leq Q''$$

  hold.

Further, since q is an infix of $(u_{\kappa+1} \ldots u_\lambda)^K$ by the assumption, the words q' and q'' are infixes of $(u_{\kappa+1} \ldots u_\lambda)^K$ as well. Then, by the induction assumption *(Lemma 5.23)*, the pairs (q', Q') and (q'', Q'') are $(\chi - 1)$-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.[10] We have proven that the pair (q, Q) is χ-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, as required. □

It remains to prove Lemma 5.22. We will proceed by induction on $h_d(p)$. To prove the base case, we will need the following lemma for $\mathsf{X} = \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

Lemma 5.24 *Let $u \in A \cup \{\varepsilon\}$ and $\mathcal{U} \subseteq \Omega_A^\omega\mathsf{M} \times 2^{\Omega_A^\omega\mathsf{M}}$ be such that, for every pair $(v, V) \in \mathcal{U}$, we have $\mathrm{Pol}_1\,\mathsf{V} \models u \leq v$ and $\mathsf{X} \models v \leq V$. Then the pair $\big(\alpha(u), \alpha(\mathcal{U})\big) \in M \times 2^M$ is ω-reducible for* $(\mathrm{Pol}\,\mathsf{V}, \mathsf{X})$.

Proof To prove this statement, it suffices to show that, for every pair $(v, V) \in \mathcal{U}$, the relation $\mathrm{Pol}\,\mathsf{V} \models u \leq v$ holds. This can be shown analogously as in the proof of Lemma 4.19.

We divide the proof into two parts depending whether $u = \varepsilon$ or $u = a \in A$.

[10] This is the point where having an arbitrary infix of the word $(u_{\kappa+1} \ldots u_\lambda)^K$ in the assumption of Lemma 5.23 is needed.

1. $u = \varepsilon$
   Let $L \in Pol(\mathcal{V})(A)$ be an arbitrary language such that $\varepsilon \in L$. By the definition of $Pol(\mathcal{V})(A)$, there exists a language $L' \in \mathcal{V}(A)$ such that $\varepsilon \in L' \subseteq L$. The relation $\mathsf{Pol}_1\,\mathsf{V} \models \varepsilon \le v$ implies that $\mathsf{V} \models \varepsilon \le v$. Hence we obtain that $v \in \overline{L'} \subseteq \overline{L}$. We have proven the desired relation $\mathsf{Pol}\,\mathsf{V} \models \varepsilon \le v$.
2. $u = a \in A$
   We proceed analogously to the previous case. Let $L \in Pol(\mathcal{V})(A)$ be an arbitrary language such that $a \in L$. By the definition of $Pol(\mathcal{V})(A)$, there exists a language $L' \in Pol_1(\mathcal{V})(A)$ such that $a \in L' \subseteq L$. Since $\mathsf{Pol}_1\,\mathsf{V} \models a \le v$, we obtain that $v \in \overline{L'} \subseteq \overline{L}$. We have proven the desired relation $\mathsf{Pol}\,\mathsf{V} \models a \le v$.

□

Now we turn back to the proof of Lemma 5.22.

Proof of Lemma 5.22 The proof goes by the induction on $h_d(p)$. If $h_d(p) = 0$, then $p \in A \cup \{\varepsilon\}$, $\nu(p) = 1$, and $\lambda(p) = 0$. Let $(q, Q) \in \mathcal{P}$ be an arbitrary pair. Then we have $\mathsf{Pol}_1\,\mathsf{V} \models p \le q$ and the pair (q, Q) is 0-factorial for $\mathsf{Pol}(\mathsf{Pol}\,\mathsf{V})^{\mathrm{d}}$, i.e., the pair $\big(\alpha(q), \alpha(Q)\big)$ is ω-reducible for $\mathsf{Pol}(\mathsf{Pol}\,\mathsf{V})^{\mathrm{d}}$. This means that there exist an ω-word $x \in \Omega_A^{\omega}\mathsf{M}$ and a set of ω-words $X \subseteq \Omega_A^{\omega}\mathsf{M}$ satisfying the following conditions:

- $\mathsf{Pol}(\mathsf{Pol}\,\mathsf{V})^{\mathrm{d}} \models x \le X$,
- $\widehat{\alpha}(x) = \alpha(q), \widehat{\alpha}(X) = \alpha(Q)$.

Since the homomorphism α is $(\mathsf{Pol}_1\mathsf{V})$-compatible, we have $\mathsf{Pol}_1\mathsf{V} \models x = q$.[11] Using the relation $\mathsf{Pol}_1\mathsf{V} \models p \le q$, we obtain $\mathsf{Pol}_1\mathsf{V} \models p \le x$. Then, by Lemma 5.24, the pair $\big(\alpha(p), \alpha(\mathcal{P})\big)$ is ω-reducible for $\big(\mathsf{Pol}\,\mathsf{V}, \mathsf{Pol}(\mathsf{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$. This completes the induction basis of the proof of Lemma 5.22.

Now we move to the induction step. Suppose that $h_d(p) \ge 1$. Let $d(p) = (p_1, \ldots, p_m)$. Then two cases can occur:

a) $m = 2$,
b) $m > 2$.

The proceeding in Case a) is straightforward. Let $d(p) = (p_1, p_2)$. For $i = 1, 2$, denote by $\mathcal{P}_i$ the maximal subset of $A^* \times 2^{A^*}$ such that, for every $(q_i, Q_i) \in \mathcal{P}_i$, the relation $\mathsf{Pol}_{\nu(p_i)}\,\mathsf{V} \models p_i \le q_i$ holds and the pair (q_i, Q_i) is $\lambda(p_i)$-factorial. Then, by the induction assumption, the pairs $\big(\alpha(p_1), \alpha(\mathcal{P}_1)\big)$, $\big(\alpha(p_2), \alpha(\mathcal{P}_2)\big)$ are ω-reducible for $\big(\mathsf{Pol}\,\mathsf{V}, \mathsf{Pol}(\mathsf{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$. We have

$$\begin{aligned}\nu(p_i) &= 1 + 3 \cdot 2^{|M| \cdot 2^{|M|}} \cdot h_d(p_i) \le 1 + 3 \cdot 2^{|M| \cdot 2^{|M|}} \cdot \big(h_d(p) - 1\big) = \\ &= 1 + 3 \cdot 2^{|M| \cdot 2^{|M|}} \cdot h_d(p) - 3 \cdot 2^{|M| \cdot 2^{|M|}} = \nu(p) - 3 \cdot 2^{|M| \cdot 2^{|M|}} < \nu(p) - 1\end{aligned}$$

and similarly

$$\lambda(p_i) < \lambda(p) - 1.$$

[11] This is the only point in the proof of the ω-reducibility of $\mathcal{P}_{\mathsf{Pol}(\mathsf{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ where the $(\mathsf{Pol}_1\mathsf{V})$-compatibility of α is needed.

Then, by Lemma 5.8, we obtain that $\mathcal{P} \subseteq \mathcal{P}_1\mathcal{P}_2$. Using Lemmas 5.12 and 5.13, we obtain that the pair $\big(\alpha(p), \alpha(\mathcal{P})\big)$ is ω-reducible for $\big(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$.

Now suppose that $h_d(p) \geq 1$ and $d(p) = (p_1, \ldots, p_m)$, where $m > 2$. Then there exists an idempotent $f \in E(M)$ such that $\alpha(p_1) = \cdots = \alpha(p_m) = \alpha(p) = f$.

To complete the proof in this case, we use Lemma 5.18. Let

$$j := \nu(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}} = 1 + 3 \cdot 2^{|M|\cdot 2^{|M|}} \cdot h_d(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}},$$

$$l := \lambda(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}} = 3 \cdot 2^{|M|\cdot 2^{|M|}} \cdot h_d(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}}.$$

Then we have

$$\begin{aligned} j &= \nu(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}} = \\ &= 1 + 3 \cdot 2^{|M|\cdot 2^{|M|}} \cdot h_d(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}} \geq \\ &\geq 1 + 3 \cdot 2^{|M|\cdot 2^{|M|}} - 2 \cdot 2^{|M|\cdot 2^{|M|}} = \\ &= 1 + 2^{|M|\cdot 2^{|M|}} > 2^{|M|\cdot 2^{|M|}} \end{aligned}$$

and similarly

$$l \geq 2^{|M|\cdot 2^{|M|}}.$$

To simplify the notation in what follows, we will write $\mathrm{i}(p_1, \ldots, p_m)$ in place of

$$\mathrm{i}_{j-2^{|M|\cdot 2^{|M|}},\, l-2^{|M|\cdot 2^{|M|}}}(p_1, \ldots, p_m).$$

We have

$$\begin{aligned} j + 2 \cdot \mathrm{i}(p_1, \ldots, p_m) &= \nu(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}} + 2 \cdot \mathrm{i}(p_1, \ldots, p_m) \leq \\ &\leq \nu(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}} + 2 \cdot 2^{|M|\cdot 2^{|M|}} = \\ &= \nu(p) \end{aligned}$$

and similarly

$$l + 2 \cdot \mathrm{i}(p_1, \ldots, p_m) \leq \lambda(p).$$

By the assumption of Lemma 5.22, for every $(q, Q) \in \mathcal{P}$, the relation $\mathrm{Pol}_{\nu(p)}\,\mathsf{V} \models p \leq q$ holds and the pair (q, Q) is $\lambda(p)$-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. By the preceding, this implies that, for every $(q, Q) \in \mathcal{P}$, the relation

$$\mathrm{Pol}_{j+2\cdot\mathrm{i}(p_1,\ldots,p_m)}\,\mathsf{V} \models p \leq q$$

holds and the pair (q, Q) is $\big(l + 2 \cdot \mathrm{i}(p_1, \ldots, p_m)\big)$-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$.

Further, for every $i \in \{1, \ldots, m\}$, let $\mathcal{P}_i$ be the maximal subset of $A^* \times 2^{A^*}$ such that, for every pair $(q_i, Q_i) \in \mathcal{P}_i$, the relation $\mathrm{Pol}_{j-2^{|M|\cdot 2^{|M|}}} \mathsf{V} \models p_i \leq q_i$ holds and the pair (q_i, Q_i) is $(l - 2^{|M|\cdot 2^{|M|}})$-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. Since

$$\begin{aligned}\nu(p_i) = 1 + 3 \cdot 2^{|M|\cdot 2^{|M|}} \cdot h_d(p_i) \leq \\ \leq 1 + 3 \cdot 2^{|M|\cdot 2^{|M|}} \cdot \big(h_d(p) - 1\big) = 1 + 3 \cdot 2^{|M|\cdot 2^{|M|}} \cdot h_d(p) - 3 \cdot 2^{|M|\cdot 2^{|M|}} = \\ = \nu(p) - 3 \cdot 2^{|M|\cdot 2^{|M|}}\end{aligned}$$

and similarly

$$\lambda(p_i) \leq \lambda(p) - 3 \cdot 2^{|M|\cdot 2^{|M|}}$$

and

$$j - 2^{|M|\cdot 2^{|M|}} = \nu(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}} - 2^{|M|\cdot 2^{|M|}} = \nu(p) - 3 \cdot 2^{|M|\cdot 2^{|M|}},$$

$$l - 2^{|M|\cdot 2^{|M|}} = \lambda(p) - 2 \cdot 2^{|M|\cdot 2^{|M|}} - 2^{|M|\cdot 2^{|M|}} = \lambda(p) - 3 \cdot 2^{|M|\cdot 2^{|M|}},$$

we obtain, for every $i \in \{1, \ldots, m\}$ and every pair $(q_i, Q_i) \in \mathcal{P}_i$, that the relation $\mathrm{Pol}_{\nu(p_i)}\mathsf{V} \models p_i \leq q_i$ holds and and the pair (q_i, Q_i) is $\lambda(p_i)$-factorial for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$. Then, by the induction assumption, the pairs

$$\big(\alpha(p_1), \alpha(\mathcal{P}_1)\big), \ldots, \big(\alpha(p_m), \alpha(\mathcal{P}_m)\big)$$

are ω-reducible for $\big(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$. By Lemma 5.18, we obtain that the pair $\big(\alpha(p_1 \ldots p_m), \alpha(\mathcal{P})\big) = \big(\alpha(p), \alpha(\mathcal{P})\big)$ is ω-reducible for $\big(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)$, as required. □

The following corollary was proven already in the author's paper [37, Theorem 5.1].

Corollary 5.25 ([37]). *For every concatenation hierarchy with a locally finite basic pseudovariety* V_0*, the pseudovariety* $\mathsf{V}_{3/2}$ *corresponding to level* $3/2$ *is* ω*-reducible.*

Proof It follows from Theorem 5.20, from the definition of the ω-reducibility of pseudovarieties, and from the relation $\mathsf{V}_{3/2} = \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V}_0)^{\mathrm{d}}$. □

Let U be an arbitrary pseudovariety. We remind of the characterization of the pseudovariety $\mathrm{Pol}\,\mathsf{U}$ by pseudoinequalities from Proposition 2.2:

$$\mathrm{Pol}\,\mathsf{U} = [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid u, v \in \overline{\Omega}_A \mathsf{M} \text{ for some } A,\ \mathsf{U} \models u \leq v]\!].$$

If we choose $\mathsf{U} = \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$, we obtain

$$\begin{aligned}\mathrm{Pol}\big(\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\big)^{\mathrm{d}} = \\ = [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid u, v \in \overline{\Omega}_A \mathsf{M} \text{ for some } A,\ \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}} \models v \leq u]\!]. \quad (31)\end{aligned}$$

The following corollary was proven already in the author's paper [37, Corollary 5.2].

Corollary 5.26 ([37]). *For a concatenation hierarchy with a locally finite basic pseudovariety* V_0*, there is the following basis of* ω*-inequalities for the pseudovariety* $\mathsf{V}_{5/2}$*:*

$$\mathsf{V}_{5/2} = [\![u^{\omega+1} \leq u^\omega v u^\omega \mid u, v \in \Omega_A^\omega \mathsf{M} \text{ for some } A,\ \mathsf{V}_{3/2} \models v \leq u]\!].$$

Proof It follows from the relations $\mathsf{V}_{3/2} = \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V}_0)^{\mathrm{d}}$, $\mathsf{V}_{5/2} = \mathrm{Pol}\big(\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V}_0)^{\mathrm{d}}\big)^{\mathrm{d}}$, Relation (31), and Corollary 5.25. □

5.7 Algorithm computing $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$

In [25], there is an algorithm computing the set $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$, where V is a locally finite selfdual pseudovariety. We explain a connection between this algorithm and our proof of the ω-reducibility of $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$.

The following definition of a set $\mathcal{A}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ is adopted from [25, Subsection 7.2], where it is described in terms of *covering* of regular languages. In the author's paper [37, Subsection 4.3], it can be found nearly in our form and the equivalence with our form is explained there [37, Remark 4.8]. Precisely our form of the definition of $\mathcal{A}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ can be found in the author's PhD thesis [36, Subsection 2.3.2].

Let $\mathcal{A}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ be the smallest set $\mathcal{A} \subseteq M \times 2^M$ such that there exists a set $\mathcal{B} \subseteq M \times 2^{M \times 2^M}$ for which the pair $(\mathcal{A}, \mathcal{B})$ satisfies the following conditions:

1. $\forall s \in M \colon \big(s, \{s\}\big) \in \mathcal{A}$,
2. $\forall (s, S) \in \mathcal{A}, \forall S' \subseteq S \colon (s, S') \in \mathcal{A}$,
3. $\forall (s_1, S_1), (s_2, S_2) \in \mathcal{A} \colon (s_1 s_2, S_1 S_2) \in \mathcal{A}$,
4. $\forall (r, \mathcal{T}) \in \mathcal{B}, \forall (e, E) \in \mathcal{T} \cap E(M \times 2^M) \colon \big(e, E \cdot \{r, 1\} \cdot E\big) \in \mathcal{A}$,

I. $\forall s \in M \colon \big(s, \big\{(s, \{s\})\big\}\big) \in \mathcal{B}$,
II. $\forall (s, \mathcal{S}) \in \mathcal{B}, \forall \mathcal{S}' \subseteq \mathcal{S} \colon (s, \mathcal{S}') \in \mathcal{B}$,
III. $\forall (s_1, \mathcal{S}_1), (s_2, \mathcal{S}_2) \in \mathcal{B} \colon (s_1 s_2, \mathcal{S}_1 \mathcal{S}_2) \in \mathcal{B}$,
IV. $\forall (e, \mathcal{E}) \in \mathcal{B} \cap E\big(M \times 2^{M \times 2^M}\big) \colon (e, \mathcal{E} \mathcal{S}_e \mathcal{E}) \in \mathcal{B}$, where

$$\mathcal{S}_e = \big\{(s, S) \in \mathcal{A} \mid [s]_{\sim^{\mathcal{A}}_{\mathsf{V}_0}} = [e]_{\sim^{\mathcal{A}}_{\mathsf{V}_0}}\big\}.$$

Theorem 5.27 *The equality* $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M] = \mathcal{A}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ *holds.*

Proof The equality follows from [25, Theorem 7.11]. More precisely, this description of the set $\mathcal{A}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ is a translation of the original algorithm from [25], where it is described in terms of *covering* of regular languages, into the terms of generalized $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$-pointlike sets. For more details, see [37, p. 118, Footnote ^{m}]. □

An alternative proof of Theorem 5.27, which does not use results of [25], can be obtained by results of this section. We explain this in more detail in the following paragraphs.

The inclusion $\mathcal{A}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M] \subseteq \mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ follows from the ω-reducibility for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ of all pairs $(s, S) \in \mathcal{A}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$. To prove the ω-reducibility for

$\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ of a pair (s, S) created by Rule 4 of the algorithm above, we need to prove also the ω-reducibility for $\left(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\right)$ of all pairs $(s, \mathcal{S})$ created by Rules I–IV of the algorithm. The ω-reducibility of all these pairs (s, S) and $(s, \mathcal{S})$ was proven by Lemmas 4.8, 4.9, 4.10, Corollary 4.14, Lemmas 5.11, 5.12, 5.13, and Corollary 5.17.

The inclusion $\mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M] \subseteq \mathcal{A}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ follows from the fact that every pair $(s, S) \in \mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ can be built by Rules 1–4 and I–IV of the algorithm above – this fact is implicitly included in the proof of Theorem 5.20. Indeed, in this proof, we showed by induction that every "non-simple" pair $(s, S) \in \mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ can be built by some "simpler" pairs ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ and then the ω-reducibility of (s, S) was derived using Lemmas 4.9, 4.10, 4.15, and 5.22. Furthermore,

- Lemma 4.15 was proven using Lemmas 4.9, 4.10 and Corollary 4.13, which show that every pair built from pairs (s, S) ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ and pairs $(s, \mathcal{S})$ ω-reducible for $\left(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\right)$ by Rules 2–4 of the algorithm is ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$,
- Lemma 5.22 shows that every pair $(s, \mathcal{S}) \in M \times 2^{M \times 2^M}$ satisfying specific conditions is ω-reducible for $\left(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\right)$. We proved this by induction. We showed that every "non-simple" pair $(s, \mathcal{S})$ satisfying given conditions can be built by some "simpler" pairs ω-reducible for $\left(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\right)$ and then the ω-reducibility of $(s, \mathcal{S})$ was derived using Lemmas 5.12, 5.13, and 5.18.
  Furthermore, Lemma 5.18 was proven using Lemmas 5.12, 5.13 and Corollary 5.17, which show that every pair built from pairs $(s, \mathcal{S})$ ω-reducible for $\left(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\right)$ and pairs (s, S) ω-reducible for $\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}$ by Rules II–IV of the algorithm is ω-reducible for $\left(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\right)$.
  We used Lemma 5.22 for the pair $\left(\alpha(\widetilde{v}), \{(e, E)\}\right)$ from the proof of Theorem 5.21 – we showed that the pair $\left(\alpha(\widetilde{v}), \{(e, E)\}\right)$ is ω-reducible for $\left(\mathrm{Pol}\,\mathsf{V}, \mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}\right)$. Then we deduced, using Lemmas 4.15, 4.9, 4.10, the ω-reducibility of the given pair $(s, S) = \left(\alpha(\gamma), \alpha(U)\right) \in \mathcal{P}_{\mathrm{Pol}(\mathrm{Pol}\,\mathsf{V})^{\mathrm{d}}}[M]$ in the case of an idempotent branching in the given factorization forest.

6 Conclusion

We have proven the ω-reducibility of pseudovarieties of ordered monoids representing levels $1/2$ and $3/2$ of concatenation hierarchies with a locally finite basic pseudovariety. The proofs are inspired by results of the paper [25] by Place on *covering* of corresponding sets of regular languages, but not using them. Our proofs can be understood as an initiation of a process of a gradual generalization of results on the ω-reducibility to higher half levels of concatenation hierarchies. As being quite difficult and technical, a potential continuation of this process is left for further work. Finally, recall the motivation for continuation of the research in this direction. Using results of the papers [6] by Almeida and Steinberg and [5] by Almeida, Klíma, Kunc, a proof of the ω-reducibility of next half levels of concatenation hierarchies with a locally finite basis would give us new results on the decidability of the membership problem for half levels of the Straubing–Thérien hierarchy.

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